# COMMUNICATIONS

## DE LA FACULTÉ DES SCIENCES DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 21 A ANNÉE 1972


# On $|\bar{N},p_n|$ Summability Factors of Infinite Series


by

RAJIV SINHA


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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

# Communications de la Faculté des Sciences de l'Université d'Ankara




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# On $|\overline{N}, p_n|$ Summability Factors of Infinite Series

RAJIV SINHA

*Department of Mathematics, Kurukshetra University, Kurukshetra. (India)*

In this paper, we have proved the following theorem:

*Theorem.* Let $\{\varepsilon_n\}$ be a sequence such that

$$\text{(i)} \quad \sum_{n=2}^{m} \frac{p_n}{P_n} |\varepsilon_n| = 0\,(1)$$

$$\text{(ii)} \quad \frac{P_n}{p_n} \triangle \varepsilon_n = 0\,(|\varepsilon_n|)$$

If

$$\sum_{v=1}^{n} |s_v| p_v = 0\,(P_n \mu_n),$$

where $\{\mu_n\}$ is a positive non decreasing sequence such that

$$\frac{P_{n+1}}{p_{n+1}} \mu_n \triangle \left( \frac{1}{\mu_n} \right) = 0\,(1), \; n \to \infty$$

then $\Sigma \dfrac{a_n \varepsilon_n}{\mu_n}$ is summable $|\overline{N}, p_n|$.

It is to be mentioned here that for $\mu_n = 1$, our theorem includes the result of Singh [Indian Jour. of Maths. Vol. 10 (1968), 19-24 ] and for $p_{n=1}$ generalizes another result of Singh [communication; De La Faculté Des Sciences De L'Science De L' Universite D' Ankara; Vol 20A. (1971),41-51].

Let $\sum_{n=1}^{\infty}$ be a given infinite series and $s_n$ its nth partial sum. Let $\{p_n\}$ be a sequence of real positive constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty.$$

We write

$$t_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \, s_v.$$

The series $\Sigma a_n$ is said to be absolutely summable $(\overline{N}, p_n)$ or summable $|\overline{N}, p_n|$, if $t_n \in BV$.

If $$\Sigma \, |s_v| \, p_v = 0 \, (P_n), \; n \to \infty,$$

then $\Sigma a_n$ is said to be bounded $[\overline{N}, p_n]$, or strongly bounded $(\overline{N}, p_n)$.

Quite recently Singh [2] has proved the following:

*Theorem* A. Let $\{\lambda_n\}$ be a convex sequence such that

$$\Sigma \frac{\lambda_n}{n} < \infty. \text{ If}$$

$$\sum_{v=1}^{n} \frac{|s_v|}{v} = 0 \, (\log n \, \mu_n),$$

where $\{\mu_n\}$ is a positive non-decreasing sequence such that

$$n \log n \, \mu_n \, \Delta \left( \frac{1}{\mu_n} \right) = 0 \, (1), \; n \to \infty,$$

then $\Sigma \dfrac{a_n \lambda_n}{\mu_n}$ is summable $|C, 1|$.

The object of this paper is to extend Theorem A for $|\overline{N}, p_n|$ summability in the form of the following theorem.

*Theorem. Let* $\{\varepsilon_n\}$ *be a sequence such that*

$$\text{(i)} \; \sum_{n=2}^{m} \frac{p_n}{P_n} \, |\varepsilon_n| = 0 \, (1)$$

$$\text{(ii)} \quad \frac{P_n}{p_n} \triangle \varepsilon_n = 0\,(\,|\,\varepsilon_n\,|\,).$$

*If*

$$\sum_{v=1}^{n} |s_v|\, p_v = 0\,(P_n\, \mu_n),$$

*where* $\{\mu_n\}$ *is a positive non-decreasing sequence such that*

$$\frac{P_{n+1}}{p_{n+1}} \mu_n \triangle \left(\frac{1}{\mu_n}\right) = 0\,(1),\ n \to \infty,$$

*then* $\Sigma \dfrac{a_n \varepsilon_n}{\mu_n}$ *is summable* $|\,\overline{N}, p_n\,|$.

It is to be mentioned here that the set of conditions on '$\varepsilon_n$' in particular case for $p_n = 1$ for all n in our theorem is lighter than the set of conditions imposed on '$\varepsilon_n$' in Theorem A. Furth ermore if we take $\mu_n = 1$ in our theorem then we have the following theorem of Singh [1 ].

*Theorem* B: If $\Sigma a_n$ is bounded $[\,\overline{N}, p_n\,]$, and $\{\varepsilon_n\}$ is a sequence satisfying the following conditions:

$$\text{(a)} \quad \sum_{2}^{m} \frac{p_n}{P_n} |\,\varepsilon_n\,| = 0\,(1),$$

$$\text{(b)} \quad \frac{P_n}{p_n} \triangle \varepsilon_n = 0\,(\,|\,\varepsilon_n\,|\,),$$

then $\Sigma a_n \varepsilon_n$ is summable $|\,\overline{N}, p_n\,|$.

*Proof of the theorem.* Let

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{i=1}^{v} \frac{a_i\, \varepsilon_i}{\mu_i}.$$

Then

$$T_n - T_{n-1} = \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) \sum_{v=1}^{n} P_{v-1} \frac{a_v\, \varepsilon_v}{\mu_v}.$$

Since

$$\sum_{v=1}^{n} P_{v-1} \frac{a_v \varepsilon_v}{\mu_v} = \sum_{v=1}^{n-1} s_v \triangle \left( \frac{P_{v-1} \varepsilon_v}{\mu_v} \right) + \frac{s_n P_{n-1} \varepsilon_n}{\mu_n},$$

therefore to prove the theorem, it is sufficient to show that

$$(1) \quad \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \frac{|s_n| \, |\varepsilon_n| P_{n-1}}{\mu_n} < \infty, \text{ as } m \to \infty,$$

and

$$(2) \quad \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left| \sum_{v=1}^{n-1} s_v \triangle \left( \frac{P_{v-1} \varepsilon_v}{\mu_v} \right) \right| < \infty, \text{ as } m \to \infty.$$

*Proof of* (1). We have

$$\sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \frac{P_{n-1}}{p_n} \frac{|\varepsilon_n| \, |s_n| p_n}{\mu_n}$$

$$= \sum_{n=1}^{m} \frac{p_n}{P_n P_{n-1}} \frac{P_{n-1}}{p_n} \frac{|\varepsilon_n| \, |s_n| p_n}{\mu_n}$$

$$\leqq \sum_{n=1}^{m-1} \mu_n P_n \left| \triangle \left( \frac{|\varepsilon_n|}{P_n \mu_n} \right) \right| + \frac{|\varepsilon_m|}{P_m \mu_m} \mu_m P_m$$

$$\leqq \sum_{n=1}^{m-1} \mu_n P_n \frac{|\triangle \varepsilon_n|}{P_n \mu_n} + |\varepsilon_{n+1}| \left[ \triangle \left( \frac{1}{\mu_n P_n} \right) \right] + 0(1)$$

$$= \sum_{n=1}^{m-1} P_n \mu_n \frac{|\triangle \varepsilon_n|}{P_n \mu_n} + \sum_{n=1}^{m-1} P_n \mu_n |\varepsilon_{n+1}| \frac{1}{P_n} \triangle \left( \frac{1}{\mu_n} \right) + \sum_{n=1}^{m-1} P_n \mu_n |\varepsilon_{n+1}| \frac{1}{\mu_n} \frac{p_{n+1}}{P_n P_{n+1}}$$

$$= \sum_{n=1}^{m-1} |\triangle \varepsilon_n| + \sum_{n=1}^{m-1} |\varepsilon_{m+1}| \frac{p_n}{P_n} + \sum_{n=1}^{m-1} |\varepsilon_{n+1}| \frac{p_{n+1}}{P_{n+1}}$$

$$\leqq \sum_{n=1}^{m-1} \frac{p_n}{P_n} |\varepsilon_n| + \sum_{n=1}^{m-1} |\varepsilon_{n+1}| \frac{p_n}{P_n} + \sum_{n=1}^{m-1} |\varepsilon_{n+1}| \frac{p_{n+1}}{P_{n+1}} = 0\,(1).$$

*Proof of* (2).

$$\sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left| \sum_{v=1}^{n-1} s_v \triangle \left( \frac{P_{v-1}\,\varepsilon_v}{\mu_v} \right) \right|$$

$$\leqq \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left| \sum_{v=1}^{n-1} \frac{s_v P_v}{\mu_v} \triangle \varepsilon_v \right| + \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left| \sum_{v=1}^{n-1} \frac{s_v\, p_v\, \varepsilon_v}{\mu_v} \right| + \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \left| \sum_{v=1}^{n-1} s_v P_v \varepsilon_v \triangle \left( \frac{1}{\mu_v} \right) \right|$$

$$= 0 \left\{ \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{v=1}^{m} \frac{|s_v|\, p_v\, |\varepsilon_v|}{\mu_v} \right\} +$$

$$+ 0 \left\{ \sum_{n=1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{v=1}^{n-1} s_v P_v \varepsilon_v \triangle \left( \frac{1}{\mu_v} \right) \Big| \right\}$$

$$= 0 \left\{ \sum_{v=1}^{m} \frac{p_v\, |s_v|\, |\varepsilon_v|}{\mu_v} \sum_{n=v+1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \right\} +$$

$$+ 0 \left\{ \sum_{v=1}^{m} s_v P_v \varepsilon_v \triangle \left( \frac{1}{\mu_v} \right) \sum_{n=v+1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \right\}$$

$$= 0 \left\{ \sum_{v=1}^{m} \frac{p_v\, |s_v|\, |\varepsilon_v|}{\mu_v P_v} \right\} + 0 \left\{ \sum_{v=1}^{m} s_v P_v \varepsilon_v \triangle \left( \frac{1}{\mu_v} \right) \sum_{n=v+1}^{m} \left( \frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \right\}$$

$$= 0\{\sum_{v=1}^{m} \frac{p_v |s_v| \; |\varepsilon_v|}{\mu_v P_v}\} + 0 \; \{\sum_{v=1}^{m} \frac{s_v P_v \varepsilon_v}{P_v} \triangle \; (\frac{1}{\mu_v})$$

$$= 0\{\sum_{v=1}^{m} \frac{p_v |s_v| \; |\varepsilon_v|}{\mu_v P_v}\}, \text{ by the given condition.}$$

Now

$$0 \; \{\sum_{v=1}^{m} \frac{p_v |s_v| \; |\varepsilon_v|}{\mu_v p_v}\} = 0\{\sum_{v=1}^{m-1} \sum_{i=1}^{v} |s_i| \, p_i | \triangle \; (\frac{|\varepsilon_v|}{\mu_v P_v})|\}$$

$$+ \; 0\{\frac{|\varepsilon_m|}{P_m \mu_m} \sum_{v=1}^{m} |s_v| \, p_v\}$$

$$= 0\{\sum_{v=1}^{m-1} |\triangle \; (\frac{|\varepsilon_v|}{\mu_v P_v}) \; | P_v \mu_v\} + 0\,(1)$$

$$= 0\{\sum_{v=1}^{m-1} \frac{|P_v \mu_v}{P_v \mu_v} |\triangle \varepsilon_v|\} + 0 \; \{\sum_{v=1}^{m-1} \frac{P_v |\varepsilon_{v+1}| \, p_{v+-1} \mu_v}{P_v P_{v+1} \mu_v}\}$$

$$+ \; 0\{\sum_{v=1}^{m-1} \frac{P_v \mu_v |\varepsilon_v|}{P_v} \triangle \; \frac{1}{\mu_v})\}$$

$$= 0\{\sum_{v=1}^{m-1} \frac{|\varepsilon_v| \, p_v}{P_v}\} + 0\{\sum_{v=1}^{m-1} \frac{|\varepsilon_{v+1}|}{P_{v+1}} p_{v+1}\}$$

$$= 0\,(1), \text{ by the hypothesis of the theorem.}$$

This completes the proof of the theorem.

*Acknowledgement.* I am grateful to Dr. N. Singh for his kind help durign the prepration of this paper.

## REFERENCES

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[2 ] ————; Absolute Summability Factors of Infinite Series, Communications: De La Faculte Des Sciences De L'Sciences De L' Universite D' Ankara, Vol. 20-A (1971); 41–51.

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Prière de s'adresser pour l'abonnement à : Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi-1973