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by

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ABSTRACT

This paper consist of two sections. In the first section which is the introduction, we have given the various examples on the non-desarguesian plane geometries. In the second section, we have tried to construct a non-desarguesian and non-pascalian real plane geometry.

I. INTRODUCTION

In general, the axioms which alignes the plane projective geometry does not imply the Desargues' and Pascal's theorems which are the fundemetal theorems of the projective goemetry. The plane which does not imply the Desargues' theorem is called "non–desarguesian". The space three dimentional and above are liable to be desarguesian. For this reason the non-desarguesian planes are mentioned. There are few examples well known in the non-desarguesian planes. We would like to refer to them briefly.

(1) In D. Hilbert's [1] non-desarguesian plane geometry, the points of the euclidean plane with the points at the infinity are taken to be points of the plane. The lines of the plane are: a) The straight lines passing through the point A (α, O), (α satisfying the condition $\alpha^2 > 2a^2 - b^2$; where a,b refers to a chosen ellipse $b^2 x^2 + a^2 y^2 = a^2 b^2$), of the euclidean plane; b) All the lines which meet the ellipse in two points P, Q with the segment of the line replaced by the arc PQ of the circle passing through A and intersecting the chosen ellipse; c) All the lines of the euclidean plane which touch the ellipse or do not meet it in the real points, with the line at infinity.

(2) In F. R. Moulton's [2] non-desarguesian plane geometry, the points are the points of the euclidean projective plane. Its lines are the straight lines of the euclidean projective plane deformed into broken lines by the equation

$$y = F(y, m)\ m\ (x - a)$$

where m and a are real numbers, and F(y, m) defined as follows:

If $m \leqq 0$, then $F(y,m) = 1$;

If $m > 0$ and $y < 0$, then $F(y,m) = 1$;

If $m > 0$ and $y \geqq 0$, then $F(y,m) = c$, $c \neq 1$, $c > 0$.

For the critical values, $m = 0$ and $a \to \infty$ the straight line

$$y = b \qquad \text{(b is real number)},$$

for $m \to \infty$ the straight line

$$x = a\ .$$

The plane geometry of F. R. Moulton is generalized by K. Levenberg [3].

(3) K. Sıtaram [4], in his real non-desarguesian plane geometry, has again all the points of the euclidean projective plane. The lines of this plane are all the straight lines except those cutting off positive intercepts on the co-ordinate axes which are rectangular. The portion of such a line intercepted between the axes is replaced by the finite arc of the parabola touching the axes at its points of intersection with the axes. The other straight lines, namely those parallel to the co-ordinate axes and those not cutting off positive intercepts, remain unchanged.

In this paper, we tried to establish, in a similar way, a non-desarguesian projective plane geometry by deforming sistematically some lines of the euclidean projective plane, and extending this deformation to the whole plane.

II. A REAL NON-DESARGUESIAN AND NON-PASCALIAN PLANE

The points of our plane are the same as the euclidean projective plane. We only deform the lines

$$\frac{x}{p} + \frac{y}{q} = 1$$

of the euclidean plane, where $p < 0 < q$, cutting the x and y axes at the points P and Q respectively, by subsituting the segment PQ of the line by the arc PQ of the ellipse

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1 .$$

This means that we have taken the lines

$$y = \begin{cases} q\left(1 - \dfrac{x}{p}\right) & \text{for } -\infty \leqq x < p,\ 0 < x \leqq +\infty \\ q\left(1 - \dfrac{x^2}{p^2}\right)^{1/2} & \text{for } p \leqq x \leqq 0 , \end{cases}$$

instead of

$$\frac{x}{p} + \frac{y}{q} = 1 .$$

The other lines of the euclidean projective plane remain unchanged.

To distinguish the lines of the euclidean plane, we will simply call them "straight lines" and the deformed ones as "lines" only.

Theorem 1. There is one and only one line joining two points of the plane.

It is enough to give the proofs for three different cases.

Case i) Both the points are in the second quadrant and the straight line joining these points has a positive slope.

Consider the points $(-x_1, y_1)$ and $(-x_2, y_2)$, where $x_1 > x_2$, $y_2 > y_1$ and $x_1, y_1 \geqq 0$, $x_2, y_2 > 0$. It is obvious that there is only one arc PQ for which the parameter p,q have the real values

$$p = -\left(\frac{y_2^2 x_1^2 - x_2^2 y_1^2}{y_2^2 - y_1^2}\right)^{1/2}, \quad q = -\left(\frac{x_2^2 y_1^2 - x_1^2 y_2^2}{x_2^2 - x_1^2}\right)^{1/2}.$$

Case ii) One of the points is in the second quadrant and the other in the first quadrant and the straight line joining these points has a positive slope.

Consider the points $(-x_1, y_1)$, (x_2, y_2), where $x_2, y_2 > 0$, $x_1, y_1 \geqq 0$, $x_1 \neq y_1$ and $y_2 > y_1$. The point (x_2, y_2) must satisfy

$$\frac{x}{p} + \frac{y}{q} = 1 \qquad (p < 0 < q)$$

and the point $(-x_1, y_1)$ must satisfy

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$

by eliminating q, we get the equation

$$(y_1^2 - y_2^2)\, p^2 - 2\, x_2\, y_1^2\, p + x_1^2\, y_2^2 + x_2^2\, y_1^2 = 0 \, .$$

As $y_2 > y_1$, this equation must have roots with opposite sign. If the negative root denoted by p_1, then the corresponding parameter q_1 will be

$$q_1 = \frac{p_1\, y_2}{p_1 - x_2} \, .$$

Case iii) One of the points is in the second quadrant and the other in the third quadrant and the straight line joining these points has a positive slope.

A similar proof, as in case ii, can be easily given.

Theorem 2. If the straight lines

$$L_1 \ldots\ldots \frac{x}{p_1} + \frac{y}{q_1} = 1 \qquad (p_1 < 0 < q_1)$$

$$L_2 \ldots\ldots \frac{x}{p_2} + \frac{y}{q_2} = 1 \; (p_2 < 0 < q_2)$$

intersect at a point which is not in the second quadrant or are parallel, the arc the ellipse corresponding to these straight lines do not intersect. If the straight lines L_1, L_2 intersect at a point in the seond quadrant, the corresponding arcs intersect at one point in the second quadrant different from the intersection of the straight lines.

The proof will be given for different cases.

Case i) Let the straight lines do not intersect at a point in the second quadrant. It is enough to take $q_2 > q_1$ and $p_1 > p_2$. But intersect at a point in the first quadrant so that $p_1 q_2 > p_2 q_1$ or $(p_2 q_2)^2 < (p_2 q_1)^2$ are satisfied. For the point of intersection of the corresponding arcs P_1Q_1 and P_2Q_2 of the ellipses, we get the point

$$K = \left[- p_1 p_2 \left(\frac{q^2_2 - q^2_1}{p^2_1 q^2_2 - p^2_2 q^2_1} \right)^{1/2}, \quad q_1 q_2 \left(\frac{p^2_2 - p^2_1}{p^2_2 q^2_1 - p^2_1 q^2_2} \right)^{1/2} \right].$$

According to the above inequalities K is not a real point. This shows that the arcs do not intersect.

If the lines intersect in the third quadrant, for this, it is enough to take $q_2 > q_1$, $p_1 > p_2$ and $p_1 q_2 < p_2 q_1$. It is easy to show that the arcs do not intersect.

Case ii) The straight lines L_1, L_2 intersect in the second quadrant, then $q_2 > q_1$ and $p_2 > p_1$ can be taken. This means that the point K becomes a real point.

Case iii) The straight lines L_1, L_2 are parallel, then $p_1 q_2 = p_2 q_1$. This shows that the point K is in infinity.

Theorem 3. If the straight line $y = - m x + n$, $m > 0$, or the straight lines parallel to the axes intersects the straight line

$$\frac{x}{p} + \frac{y}{q} = 1 \qquad (p < 0 < q)$$

at a point in the second quadrant, it also intersects the arc PQ of the ellipse.

It is easy to give the proof by considering the condition of intersection $pm \leqq n \leqq q$, and the point of the intersection being in the second quadrant.

The last two theorems prove that two lines intersect at one point only.

In order to show that our projective plane is non-desarguesian, first let us introduce the Desargues' theorem in the euclidean plane, and then observe that it does not hold in our plane.

Desargues' Theorems. 1- If the vertices of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ which are in the same plane are perspective with respect to a point S of the plane, Then the extensions of the corresponding sides intersect on a straight line.

2- If the extension of the corresponding sides of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ which are in the same plane intersect on a straight line, then the corresponding vertices of the triangles are perspective with respect to a point in the same plane.

Let the vertices A_1, A_2, B_1, B_2, C_1, C_2 of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ be in the third quadrant and the pairs of the straight lines A_1B_1, A_2B_2; B_1C_1, B_2C_2; C_1A_1, C_2A_2 be parallel. Let the straight lines B_1B_2, C_1C_2 have negative slope, but the line A_1A_2 has a possitive slope and its x–intercept is negative, y–intercept positive. In this case, the euclidean projective plane being desarguesian the straight lines A_1A_2, B_1B_2, C_1C_2 meet at a point S. If the same configuration is considered in this projective plane although the corresponding sides intersect on the line at infinity, A_1A_2 does not pass through the point of intersection of the lines B_1B_2 and C_1C_2 (Fig. 1). This shows that our plane is not desarguesian.

The Extension of The Deformation to The Entire Plane.

Constructing the non-desarguesian plane both D. Hilbert and F. R. Moulton and also the other mathematicians have have imposed different properties on some of the straight lines

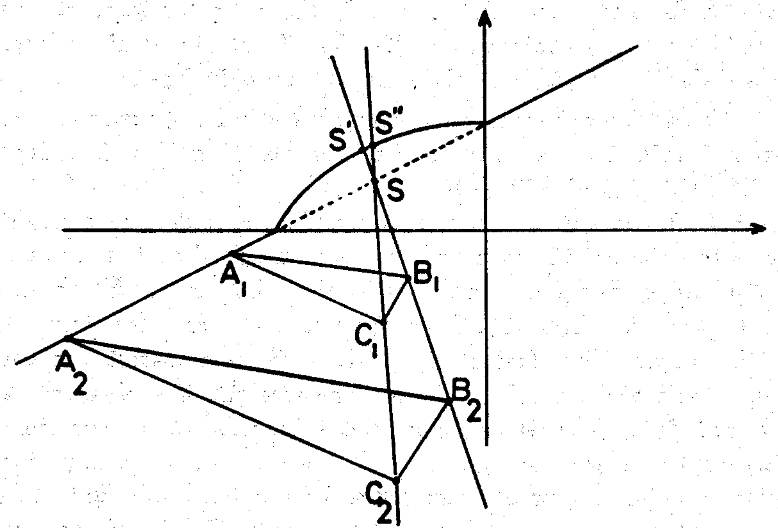

Fig. 1

of the euclidean plane. The plane that we have constructed is a new example. The deformation that we have imposed on the straight lines having negative x–intercept and positive y-intercept will be extended to the other straight lines by replacing the segments of the straight lines

$$\frac{x}{p} + \frac{y}{q} = 1 \qquad (p, q \neq 0)$$

which are in between the axes by the arcs of ellipses

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1 .$$

Thus, all the straight lines of the euclidean plane are deformed with the exception of the straight lines corresponding to the critical values of p and q.

After the extension of the deformation to the entire plane Th. 1, Th. 2 and Th. 3 are still valid.

Pascal's Theorem (*). If the points T_1, T_2, T_3 are situated on

* This theorem is due Pappus, at the same time, since it is a particular case of a theorem of Pascal it is called Pascal Theorem

the straight line t and the points S_1, S_2, S_3 on the straight line s, then points of intersections K_3, K_2, K_1 of the straight lines T_1S_2, T_2S_1; T_1S_3, T_3S_1; T_2S_3, T_3S_2 are on a straight line k.

We will show that this theorem which is an important theorem for the projective planes is not valid in our plane. So this plane is also a non-pascalian plane.

Let us consider the configuration Fig. 2 such that straight line s coincides with the x-axis and the straight line t is situated in the upper half-plane and parallel to s. Let the T_1 be on the y–axis and the points T_2, T_3 in the first quadrant, the point S_2 at the origin. The straight line T_2S_3 is choosen parallel to the y–axis and the point T_3 such that the straight line K_1K_3 has a positive slope. If this configuration is considered in our non-desarguesian plane the line T_1S_3 no longer passes through the point K_2. The point of intersection of the line S_1T_3 and T_1S_3 is K'_2. According to the theorem. 1 the points K_1, K'_2, K_3 are not on the same line. This proves the required result.

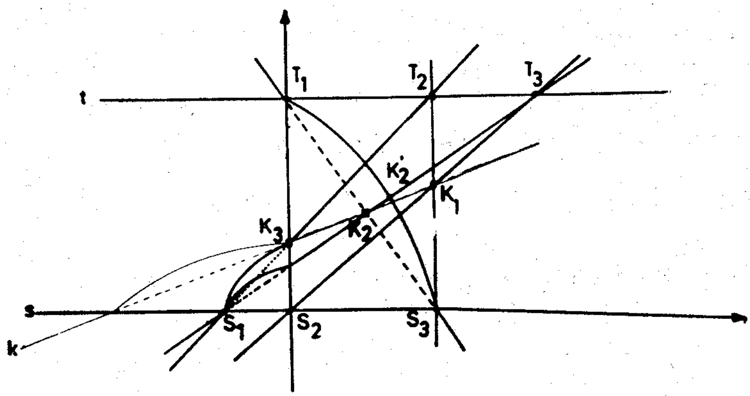

Fig. 2

According to the following definition this plane can be taken as an affine plane.

Definition. Two lines not intersecting at a finite point is called parallel.

Theorem 4. There is one and only one line parallel to a given line passing through a given point, in this plane.

The proof can easily be deduced.

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ÖZET

Bu çalışmada, D. Hilbert [1], F. R. Moulton [2] ve K. Sıtaram [4] tarafından kurulan reel non-desarguesian düzlemlerin bir yenisi, öklidyen projektif düzlemin doğrularında sistematik deformasyonlar yapmak süre*iyle, kurulmuştur.

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