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On The Constant a

by

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On The Constant a

C. ULUÇAY

De La Faculté Des Sciences De L'Université D'Ankara

SUMMARY

Let S be the class of functions

$$w = f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

analytic and schlicht in the unit disc $|z| < 1$. Let r_f denote the least upper bound of the radii of all open discs contained in the map of $w = f(z)$ that is schlicht for $|z| < 1$ and has $f'(0) = 1$. Then

$$a = \min_{f \varepsilon S} r_f$$

is the Landau constant. It is shown that $a_3 = 0$. As an immediate consequence $a > .629$

Let S be the class of functions

$$w = f(z) = z + a_2 z^2 + a_3 z^3 + \dots,$$

analytic and schlicht in the unit disc $|z| < 1$. For the purpose of obtaining a lower bound on a (defined below), Landau [1] considered a subclass T for which $|f'(z)|(1 - |z|^2) \leqq 1$. He showed that $a_2 = 0$, $|a_3| \leq 1/3$. In the present paper we reobtain Landau's results by using the definition of a Bloch function of the third kind introduced by R. M. Robinson [2] and Marty's variation [3], while $a_3 = 0$ follows from geometric considerations and an interior variation due to Schiffer [4,5]. $a_3 = 0$ has been conjectured earlier by the author. It simplifies and yields at the same time a correct proof of the lower bound. In the opinion of this writer it is hopeless to obtain the bound under consideration without first proving that $a_3 = 0$. In addition to this, the important symmetry condition $F = 0$ (Theorem 4) is also derived. Namely, we conclude with the remarkable characterization that there is a schlicht disc of radius a, lying in the map of a Bloch function of the third kind, centered at the origin, and such that its size is determined by a unique triplet of boundary points which lie at the vertices of an equilateral triangle inscribed in the circle.

Definition. Let r_f denote the least upper bound of the radii of all open discs contained in the map of $w = f(z)$ that is schlicht for $|z| < 1$ and has $f'(0) = 1$. Then

$$a = \min_{f \varepsilon S} r_f,$$

is the Landau constant.

Definition. If the map of $w = f(z)$ contains no schlicht open disc of radius greater than a, then f is called a Bloch function of the third kind.

In the following we shall use the abbreviation B. F. for a Bloch function of the third kind.

Theorem 1. Let $w = f(z)$ be a B. F. Then $a_2 = 0$, $|a_3| \leqq 1/3$.

Proof. Following Marty [3], set

$$f((z + \varepsilon)/(1 + \bar{\varepsilon}z)) - f(\varepsilon) = f'(\varepsilon)(1 - |\varepsilon|^2)\, z + \dots .$$

Here the variation used is the translation $w^* = w - f(\varepsilon)$, and the corresponding mapping function is

$$f^*(z) = f'(\varepsilon)\,(1 - |\varepsilon|^2)\, z + \dots .$$

We have

$$f'(\varepsilon)\,(1 - |\varepsilon|^2) = 1 + 2a_2\varepsilon + o(\varepsilon).$$

Since in the translation the least upper bound is preserved and that $\arg \varepsilon$ is arbitrary, it follows that $a_2 = 0$.

Next, calculating higher terms, i.e.,

$$f'(\varepsilon)\,(1 - |\varepsilon|^2) = 1 + 3a_3\varepsilon^2 - |\varepsilon|^2 + 0(\varepsilon^3),$$

it follows from the same argument that $|a_3| \leqq 1/3$.

Lemma 1. Let w_1, w_2, w_3, be three distinct points on the complex w plane. Let A_w be the area of the disc C whose circumference c contains these three points. Then A_w is given by the formula

$$A_w = \frac{\pi \mid w_2 - w_3 \mid^2}{2 - 2 \operatorname{Re} \dfrac{w_1 - w_3}{\overline{w_1 - w_3}} \cdot \overline{\dfrac{w_1 - w_2}{w_1 - w_2}}}$$

Proof. Let r_w be the radius of the disc C. Then by elementary geometry

$$r_w = \frac{\mid w_2 - w_3 \mid}{2 \sin \theta} .$$

Here

$$\theta = \sphericalangle\ (w_1 w_2,\ w_1 w_3),\ 0 < \theta < \pi .$$

Hence

(1)
$$A_w = \frac{\pi \ |w_2 - w_3|^2}{4 \sin^2 \theta} .$$

Now, put

$$W = \frac{w_1 - w_3}{w_1 - w_2} .$$

Then

$$\frac{w_1 - w_3}{\overline{w_1 - w_3}} \cdot \overline{\frac{w_1 - w_2}{w_1 - w_2}} = W \ / \ \overline{W} .$$

But

$$\theta = \arg W,$$

and so

$$W / \overline{W} = e^{2i\theta} = \cos 2\theta + i \sin 2\theta$$

Hence,

$$\cos 2\theta = \operatorname{Re}(W / \overline{W}) = \operatorname{Re} \frac{w_1 - w_3}{\overline{w_1 - w_3}} \ \overline{\frac{w_1 - w_2}{w_1 - w_2}} .$$

Carrying the value of $\cos 2\theta$ in formula (1) we obtain the desired result.

Lemma 2. Consider the transformation

(2)
$$w^* = w + \varepsilon / w$$

where ε is real and sufficiently small. Let w_i^* be the image of w_i, $i = 1, 2, 3$. Let C_w^* be the disc whose circumference c^*_w contains the points w_i^*, $i = 1, 2, 3$, and A_w^* its area. Then

$$A_w^* = A_w\,(1 + 2\,\varepsilon\,\mathrm{Im}F) + 0(\varepsilon^2),$$

where

$$F = \frac{\cot\theta}{w_1 w_3} - \frac{\cot\theta}{w_1 w_2} - \frac{i}{w_2 w_3}\ .$$

Proof. The area A_w^* of the disc C_w^* whose circumference c_w^* contains the points w_i^*, $i = 1, 2, 3$ is given by the formula of Lemma 1, i.e.,

$$(3)\qquad A_w^* = \frac{\pi\ |\,w_2^*—w_3^*\,|^2}{2—2\mathrm{Re}\ \dfrac{w_1^*—w_3^*}{\overline{w_1^*—w_3^*}}\ \dfrac{\overline{w_1^*—w_2^*}}{w_1^*—w_2^*}}$$

Setting (2) into (3), we obtain

$$A_w^* = A_w - 2\pi\varepsilon\left(\mathrm{Im}\left(\frac{1}{w_1w_3} - \frac{1}{w_1w_2}\right)\mathrm{Re}\ i(W/\overline{W})\right.$$

$$\left. + (1—\mathrm{Re}\ W/\overline{W})\ \mathrm{Re}\ \frac{1}{w_2w_3}\right)\ \frac{|\,w_2—w_3\,|^2}{(1—\mathrm{Re}\ W/\overline{W})^2} + 0(\varepsilon^2),$$

from which the formula follows.

In connection with (2) we recall the following variational formula due to Schiffer [4]. Let R be the map of $f\varepsilon$S. Denote by Γ the boundary of R. Let w_0 be a fixed point in R and consider the transformation

$$(4)\qquad w^* = w + \frac{\varepsilon}{w\ —w_0}$$

where ε is a complex number. If $|\,\varepsilon\,|$ is sufficiently small, then $w^*(w)$ is univalent on Γ and maps it into Γ^* that is the boundary of a new simply connected region [4,5]. The function $f^*(z)$ which maps the unit disc $|\,z\,| < 1$, conformally onto R* is given by

$$f^*(z) = f(z) + \varepsilon\left\{\frac{zf'(z)}{z_0f'(z_0)^2(z_0—z)} + \frac{1}{f(z)—f(z_0)} + \frac{f'(z)}{f(z_0)}\right\}$$

$$+\ \bar{\varepsilon}\left\{\frac{z^2f'(z)}{\bar{z}_0\overline{f'(z_0)}^2(1—\bar{z}_0z)} — \frac{z^2f'(z)}{\overline{f(z_0)}}\right\} + o(\varepsilon),\ \ |\,z_0\,|<1.$$

The leading coefficient a_1^* in the expansion of f^* about the origin is obtained by calculating

$$\lim_{z \to 0} \frac{f^*(z)}{z},$$

Namely,

$$a_1^* = 1 + \varepsilon \left\{ \frac{1}{z_0^2 f'(z_0)^2} - \frac{1-2a_2 f(z_0)}{f(z_0)^2} \right\} + o(\varepsilon) \tag{5}$$

Putting in (5), $a_2 = 0$ and letting $z_0 \to 0$, we find

$$a_1^* = 1 - 4a_3 \varepsilon + o(\varepsilon) \tag{6}$$

From here on we shall assume $\varepsilon > 0$, and unless stated otherwise f will denote a B. F.

Further, we introduce through the one-parameter family of B. F., i. e., $e^{i\varphi} f(e^{-i\varphi} z)$, a real parameter φ, $0 \leq \varphi \leq 2\pi$, corresponding to the rotations φ of the complex w-plane. Then R^*, f^*, a_1^* should read respectively $R^*(\varphi)$, $f^*(z, \varphi)$, $a_1^*(\varphi)$ with

$$a_1^*(\varphi) = 1 - 4a_3 \varepsilon\, e^{-2i\varphi} + o(\varepsilon),$$

where $o(\varepsilon)$ is uniform with respect to φ. For each φ

$$r_f^*(\varphi), \quad r_F^*(\varphi) = \frac{r_f^*(\varphi)}{|a_1^*(\varphi)|},$$

will denote the l. u. b. of the radii of all schlicht discs contained in $R^*(\varphi)$ and in the map of

$$F^*(z, \varphi) = f^*(z, \varphi) / a_1^*(\varphi)$$

respectively.

Lemma 3. If f is a B. F., then $r_f^*(\varphi) - a$ has a constant sign for all φ, as $\varepsilon \to 0$.

Proof. We observe that for fixed φ, ε determines uniquely R^* and the map of F^*, and thereby r_f^* and r_F^* respectively. Hence for fixed φ there is a one to one continuous correspondence between r_F^* and r_f^* and for $\varepsilon = 0$

$$r_f^* = r_F^* = a$$

Further, for convenience we set

$$s_f^* = r_f^* - a, \quad s_F^* = r_F^* - a$$

and let

$$t^*(\varphi) = s_F^*(\varphi) - s_f^*(\varphi)$$

since $s_F^* \geqq 0$ for all φ, then for sufficiently small values of s_F^*, the representative curve $\Gamma\varphi$ of the pairs (s_F^*, s_f^*), resulting from the one to one continuous correspondence, for each fixed φ, between r_F^* and r_f^*, lies in the right half-plane $s_F^* \geqq 0$. Now, $t^*(\varphi)$, for $\varepsilon \geqq 0$, vanishes only on the half-line defined by $s_f^* = s_F^*$ in which case $\Gamma\varphi$ may reduce to a point, namely, the origin. Hence, outside of this line, for $\varepsilon > 0$; $t^*(\varphi) < 0$ with $\Gamma\varphi$ lying in the angle determined by the half-lines $s_f^* \geqq 0$, $s_f^* = s_F^*$, and $t^*(\varphi) > 0$ with $\Gamma\varphi$ lying in the suplementary angle defined by the half-lines $s_f^* = s_F^*$, $s_f^* \leqq 0$. Next, if there exists a curve $\Gamma\varphi$ lying in an angle arbitrarily small with vertex at the origin and containing the s_F^* - axis, then $f^*(z, \varphi)$ being compact and therefore $\Gamma\varphi$ continuous, it follows that for $\varepsilon > 0$, there exists $\Gamma\varphi$ lying on the s_F^* axis. This is impossible, since for this $\Gamma\varphi$, $s_f^* = 0$, while in view of $t^*(\varphi) > 0$, $s_F^* > 0$. Same contradiction is reached for a curve $\Gamma\varphi$ lying on the s_f^* - axis. Hence, in view of the continuity of $\Gamma\varphi$ with respect to φ, it follows that the curves $\Gamma\varphi$ for all φ, either lie in a subangle of the angle defined by the $s_f^* > 0$ axis and s_F^* axis, or in a subangle of the angle defined by the $s_f^* < 0$-axis and s_F^*-axis.

Lemma 4. Let $f \varepsilon S$. Suppose that a is the l. u. b. of all schlicht discs contained in the map of f. Then f assumes the interior of at least one schlicht circle of radius a lying at finite distance.

Proof. Suppose that in the map R of f, there are no schlicht discs of radius a lying at finite distance. There are two possible cases. (i) R is the whole plane such that every point of the plane is either an interior point or a boundary point of R. Then there is at least one *end* point e of a boundary continuum γ. By suppressing a subcontinuum containing e, one may increase the inner radius of R without increasing the value of a. (ii) In the alternative case, there is a boundary arc γ, such that it is possible to replace a part of γ by an arc lying outside of R so as to increase the inner radius of R without increasing the value of a. These contradictions establish the lemma.

Corollary. If f is a B. F., then the map of f contains at least one schlicht disc of radius a lying at finite distance.

Lemma 5. Let f be a B. F. For each fixed φ, F^* (z, φ) assumes univalently the interior of at least one circle $C_F{}^*$ of radius $r_F{}^*$, lying at finite distance.

Proof. Assuming the contrary, F^* (z,φ) for φ fixed, assumes univalently the interior of a circle C^*_∞ of radius $r_F{}^*$ lying at infinite distance. This implies the existence of a circle C_∞ of radius a lying at infinite distance in R and such that $C^*_\infty \rightarrow C_\infty$ as $\varepsilon \rightarrow 0$. But since (2) preserves the size of a circle lying at infinity, it follows that $r_f{}^* = a$, Then by the one to one continuous correspondence, for a fixed φ, between $r_F{}^*$ and $r_f{}^*$ we have $r_F{}^* = a$, and the conclusion is reached in view of Lemma 4.

Lemma 6. Let f be a B. F., and R* the map of f^*. Let $C_f{}^*$ denote a schlicht disc of radius $r_f{}^*$ lying in R* at finite distance. Then formula of Lemma 2 holds, up to an error term of higher order.

Proof. Suppose that on the circumference $c_f{}^*$ of $C_f{}^*$ lie three boundary points (omitted values) $w_i{}^*, i = 1, 2, 3$. Since, as $\varepsilon \rightarrow 0$, the circumference $c_f{}^*$ tends to the circumference c of C with radius a, then $w_i{}^* \rightarrow w_i$ lying on c. Let c^* be the image of c under (2). $c_f{}^*$ intersects c^* at the neighboring points $\widetilde{w_i}^*$ of $w_i{}^*$ (it is possible that some or all of the points $\widetilde{w_i}^*$ coincide with $w_i{}^*$). Consequently, the images $\widetilde{w_i}$ of $\widetilde{w_i}^*$ under (2) being on c, the formula of Lemma 2 is applicable to the disc $C_f{}^*$ whose circumference $c_f{}^*$ contains the points $\widetilde{w_i}^*$, and whose area is $A_f{}^*$. On the other hand, since as $\varepsilon \rightarrow 0$, the points $\widetilde{w_i} \rightarrow w_i$, we may write

$$\widetilde{w_i} = w_i + \varepsilon_i, \qquad i = 1, 2, 3,$$

where $\varepsilon_i \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence formula of Lemma 2 holds, up to an error term of higher order. Namely,

$$A_f{}^* = \pi a^2 (1 + 2\varepsilon \operatorname{Im} F) + o(\varepsilon). \tag{7}$$

It is possible that the size of $C_f{}^*$ may be fixed by only two boundary points w_1^*, w_3^*, such will be the case, e. g., of C_f^* rolling along and between two parallel arcs which are parts of the boundary of R*, then the foregoing argument is still applicable, and formula (7) is valid, except that w_2 ($\neq w_1, w_3$) is any fixed point on c.

Lemma 7. Introducing the parameter φ, we may write for a fixed F,

(8) $$r_f^*(\varphi) = a\,(1 + \varepsilon \operatorname{Im} e^{-2i\varphi} F) + o(\varepsilon)$$

and

(9) $$r_F^*(\varphi) = a\,(1 + \varepsilon \operatorname{Im} e^{-2i\varphi}\,(F + 4ia_3)) + o(\varepsilon)$$

Proof. Indeed, assuming, say, r_f^* $(\varphi) \geqq a$ (lemma 3) we know (lemma 5) that to each φ it can be associated in R*(φ) a disc $C_f^*(\varphi)$ with radius r_f^* (φ) given by the formula

$$r_f^*(\varphi) = a\,(1 + \varepsilon \operatorname{Im} e^{-2i\varphi} F\varphi) + o(\varepsilon)$$

where $F\varphi$ is the symmetric function associated with the disc $C\varphi$ in R, with radius a, and such that $C_f^*(\varphi) \to C\varphi$ as $\varepsilon \to 0$. For the purpose of identifying $C\varphi$ with $F\varphi$, it is important to think of $F\varphi$ as a form rather than as a number. Accordingly, for the identification purpose it is then preferable to write $F\varphi$ in the equivalent symmetric form

$$F\varphi = -\,i\;\frac{\sum\limits_{j=1}^{j=3} w_j \tan\,\theta_j}{w_1 w_2 w_3 \sum\limits_{j=1}^{j=3} \tan\,\theta_j}\,.$$

Next, we note that for each φ the disc $C\varphi$, and thereby C_f^* (φ), is uniquely determined within the totality of the discs of radius a in R, as soon as $\operatorname{Im} e^{-2i\varphi}\ F\varphi$ is minimum, with $|\operatorname{Re} e^{-2i\varphi}\ F\varphi\ |$ minimum, and, say, $\operatorname{Re} e^{-2i\varphi}\ F\varphi > 0$, whenever there are two $F\varphi$'s with the above specifications. Hence, under these specifications for $F\varphi$, we conclude that there is a one to one correspondence between C_f^* (φ) and $C\varphi$ as $\varepsilon \to 0$, and that $C_f^*(\varphi)$, now uniquely defined by r_f^* (φ) is continuous in φ , so is $C\varphi$. In conclusion, as φ varies from 0 to 2π, $C\varphi$ (or $F\varphi$) either remains fixed (or constant), or else, rolls continuously along two non intersecting parallel boundary continua lying at finite distance. The simple closed curve described by the center of $C\varphi$ and lying in the interior of R, contains one of the two continua which may consist of a single point. This contradicts however, the simply connectivity of R. So $F\varphi$ must be constant.

Theorem 2. If f is a B. F., then $F = 0$.

Proof. From relation (8) we can write

$$s_f^* = a \, \varepsilon \, \mathrm{Im} \, e^{-2i\varphi} \, F + o(\varepsilon).$$

Since by Lemma 3, $s_f^*(\varphi)$ has a constant sign for all φ, it is clear that

$$F = 0. \tag{10}$$

Theorem 3. If f is a B. F., then $a_3 = 0$.

Proof. From relation (9) we can write

$$s_F^*(\varphi) = a \, \varepsilon \, \mathrm{Im} \, e^{-2i\varphi} \, (F + 4ia_3) + o(\varepsilon)$$

Since $s_F^*(\varphi) \geqq 0$ for all φ, it follows that

$$F + 4ia_3 = 0$$

In view of Theorem 2, we have

$$a_3 = 0$$

One verifies at once that (10) holds for w_1, w_2, w_3 lying at the vertices of an equilateral triangle, centre at the origin. In fact we can state the following remarkable theorem.

Theorem 4. $F = 0$ holds if and only if w_1, w_2, w_3 lie at the vertices of an equilateral triangle, centre at the origin.

Proof. First we remark that $F = 0$ is not affected when the w plane undergoes an homothetic transformation or a rotation or both. Now, $F = 0$ can be written as

$$\tan \theta = \frac{i}{w_1} (w_3 - w_2). \tag{11}$$

We recall that θ is the angle at the vertex w_1. Applying the foregoing transformations so that w_1 goes over into i, then w_2, w_3 will go respectively over, say, ω_2, ω_3. Accordingly (11) takes the form

$$\tan \theta = \omega_3 - \omega_2.$$

This last equation implies that the vector $\omega_2\omega_3$ must be parallel to the real axis. Similarly, should we have defined θ to be the angle at the other vertices, we would end up with the same conclusion. But the only triangle whose sides are successively parallel to the real axis when the w plane undergoes the foregoing transformations so as to take respectively the vertices into i, is evidently the equilateral triangle with centre at the origin.

As an immediate consequence of Theorem 3 we have

Theorem 5. $a > .629$.

Proof. Suppose that $f(z)$ is a B. F. and omits the value c. The function

$$F(z) = \frac{cf(z)}{c - f(z)} = z + b_2 z^2 + b_3 z^3 + \dots$$

where $b_2 = 1 / c$, $b_3 = 1 / c^2$, is analytic and schlicht for $|z| < 1$. Goluzin's inequality

$$|b_3 - \alpha b_2^2| \leq 2 e^{\frac{-2\alpha}{1-\alpha}} + 1,$$

for all $0 < \alpha < 1$, takes the form

$$\frac{1 - \alpha}{|c|^2} \leq 2 e^{\frac{-2\alpha}{1-\alpha}} + 1.$$

For the particular value $\alpha = .45$, we obtain

$$|c| > .629$$

If c is nearest to the origin then $a \geq |c|$. Hence $a > .629$.

Remark. It should be noticed that $a_3=0$ and the lower bound thereof would follow at once from a known property of a B. F., [6], i. e., R is the whole plane such that every point of the plane is either an interior or a boundary point of R. Indeed, since (2) preserves the size of a schlicht disc of radius a lying at infinity in R, the corresponding radius of the disc at infinity in the map of $F^*(z, \varphi)$ is $a/|a_1^*(\varphi)|$. But $r_F^*(\varphi) \geq a$, and à fortiori $|a_1^*(\varphi)| \leq 1$, or $\mathrm{Re}\, e^{-2i\varphi} a_3 \geq 0$, i. e., $a_3 = 0$. This follows also from (9) where $F=0$ *as* $w \to \infty$.

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ÖZET

$|z| < 1$ dairesinde

$$w = f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

analitik ve schlicht fonksiyonlarının S sınıfını gözönüne alalım. $w = f(z)$ tasvirinin ihtiva ettiği bütün schlicht açık dairelerin yarıçaplarının en küçük üst sınırını r_f ile gösterelim.

$$a = \min_{f \varepsilon S} r_f$$

Landau sabitidir. Gösteriliyor ki, $f \varepsilon S$ üçüncü nevi bir Bloch fonksiyonu ise, bu takdirde $a_3 = 0$. Bunun bir neticesi olarak $a > .629$.

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