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General Dual Motion of n Moving Reference Frames*

by

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I. Abstract:

We consider a kinematical system of n Euclidean 3-dimensional spaces $S_i (i = 1,2,3,...,n)$ moving with respect to each other and containing a differentiable line-system of one dual parameter $\tau = t + \varepsilon t^*$. The case of $t^* = 0$ is considered as a special case. In sections II and III, for the analysis of the relative motion of the system we derive the properties of general dual motions in matrix algebra over the ring of dual numbers. In section IV, this procedure enables us to obtain, more elegantly, the generalizations for the expressions obtained by Blaschke and Müller [1], [2], [3], Sherby and Chmielewski [4] and Janik [8]. Our resulting expressions involve only the components of the pfaffian vector and their local derivatives. Hence in section V we give them a geometrical interpretation and finally we generalize the configuration of the instantaneous pole points (centros) and pole lines (polgeraden) of a planar system to the configuration of instantaneous helicoidal axes of a spatial kinematical system of n moving reference frames.

II. General Dual Motion.

In the Euclidean 3-dimensional space R^3, lines combined with one of their two directions can be represented by unit dual vectors over the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented

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lines in R^3 are in one-to-one correspondence with the points of a dual unit sphere D^3 [5]. A dual point on D^3 corresponds to a line in R^3, two different points on D^3, in general, respresent two skew-lines in R^3. A differentiable curve on D^3 represents a ruled surface (regulus) in R^3, i.e. a ruled surface is determined by a dual unit vector, a function of one real variable:

$$\vec{A} = \vec{A}(t), \vec{A}^2 = 1 \tag{1}$$

The lines $\vec{A}(t)$ are the generators of the ruled surface. This kind of definition of surfaces, in some sense, is more general than that based on points, since quite regular curves on the dual sphere may represent ruled surfaces with complicated point singularities in R^3.

In general, a dual unit vector, a function of one *dual variable* $\tau = t + \varepsilon t^*$, $\varepsilon^2 = 0$:

$$\vec{A}(\tau) = \vec{A}(t) + \varepsilon\, t^* \dot{\vec{A}}(t), \ \vec{A}^2 = 1, \ \dot{\vec{A}} = \frac{d\vec{A}}{dt} \tag{2}$$

is a *differentiable line-system** (synektisches Strahlensystem) of one dual variable "related to" the regulus (ruled surface) $\vec{A}(t)$.However, ruled surfaces are not the line-systems with the closest analogy to spherical curves. A better analogy can be obtained from differentiable line-systems of one dual parameter [6]. A differentiable function of a dual variable can be defined by analogy with a *complex variable.* A differentiable function of a dual variable has the form

$$F(\tau) = F(t) + \varepsilon\, t^* \dot{F}(t), \ \dot{F}(t) = \frac{dF(t)}{dt}, \ \tau = t + \varepsilon\, t^*, \ \varepsilon^2 = 0. \tag{3}$$

e.g. $\cos\tau = \cos t - \varepsilon\, t^* \sin t$. Formulas for differentiation and integration of $F(\tau)$ are

* A differentiable line-system is a very particular line-congruence with two real parameters t and t*.

$$\left.\begin{array}{l} \dfrac{dF(\tau)}{d\tau} \equiv \dot{F}(\tau) = \dot{F}(t) + \varepsilon\, t^* \ddot{F}(t) \\ \text{(in particular, for } \tau = t + \varepsilon\, 0\text{: } \dot{F}(\tau) = \dot{F}(t)) \\ \displaystyle\int_{\tau_0}^{\tau} F(Y)\, dY = \int_{t_0}^{t} F(y)\, dy + \varepsilon\,(t^* F(t) - t_0^* F(t_0)) \end{array}\right\} \quad (4)$$

where $Y = y + \varepsilon\, y^*$, $\tau_0 = t_0 + \varepsilon\, t_0^*$.

The Euclidean 3-dimensional line-geometry, expressed with the help of dual unit vectors, is therefore closely analogous to the spherical geometry, expressed with the help of real unit vectors. Properties of elementary spherical geometry can be carried over to line-geometry by some simple translation rules [6]. Distance, angle and orientation of two directed lines $\vec{A}$ and $\vec{B}$ $(\vec{A}^2 = \vec{B}^2 = 1)$ are determined by the inner product

$$\operatorname{Cos} \Theta = \vec{A}.\vec{B},\ \Theta = \theta + \varepsilon\, \theta^* = \text{angle} + \varepsilon \text{ distance}, \quad 0 \leqq \theta \leqq \pi.$$

Since a Euclidean motion in R^3 leaves the angle and the distance between two lines unchanged it will also leave the dual angle between two lines unchanged. Therefore the corresponding transformation in D^3 will leave the inner product

$$\vec{A}.\vec{B} = A\ B^T$$

invariant, where A and B are column matrices. Hence the dual spherical motion, expressed with the help of dual unit vectors, is closely analogous to the real spherical motion [2], expressed with the help of real unit vectors. When the center of the dual unit sphere must remain fixed the transformation group in D^3, the image of the Euclidean motions in R^3, does not contain any translations. Hence the following theorem is known [5]:

Theorem 1. The Euclidean motions in R^3 are represented in D^3 by the dual 3×3 orthogonal matrices $A = (A_{ij})$: $AA^T = E$, A_{ij} dual numbers and E 3×3 unit matrix.

The Lie algebra $L(O_D{}^3)$ of the group of 3×3 orthogonal dual matrices is the algebra of skew-symmetric 3×3 dual matrices. This is seen by differentiation of $AA^T = E$ with respect to τ. Therefore, we can easily extend all known formulas about real spherical motions. But it is necessary to pay attention to the zero divisors, because no number εt^*, of the dual number $\tau = t+\varepsilon t^*$, has an inverse in the algebra; but the other laws of the algebra of dual numbers are the same as the laws of the algebra of complex numbers $(t+it^*,\ i^2 = -1)$.

Since the oriented lines in R^3 are in one-to-one correspondence with the points on a dual unit sphere D^3, we take one dual unit sphere for each one of n rigid bodies $S_1, S_2, \dots, S_n$; such that oriented lines in S_f are in one-to-one correspondence with the points on the dual unit sphere $D_f{}^3$ $(f=1,2,3,\dots,n)$. These dual unit spheres have a common center 0 and are in motion relative to each other. In this paper, this motion is considered as *one-dual parameter spherical motion.*

The coordinate system $|0_i;\ \vec{r}_{i1}, \vec{r}_{i2}, \vec{r}_{i3}|$, $(i=1,2,\dots,n)$ is right-handed orthonormal system which represents the space S_i. The corresponding dual orthonormal coordinate axes are

$$\vec{R}_{ij} = \vec{r}_{ij} + \varepsilon\ \vec{r}^*_{ij} \quad (i=1,2,\dots,n \text{ and } j=1,2,3)$$

i.e. $R^2_{ij} = 1$, $\vec{r}_{ij}$ represents the direction of the axis, and $\vec{r}_{ij}{}^*$ is the moment of a unit vector in the directed axis, with respect to a fixed origin point 0' in the space, that is

$$\vec{r}_{ij}{}^* = \overrightarrow{0'0_i} \times \vec{r}_{ij} \quad .$$

Hence the corresponding dual *pfaffian forms* are

$$\Omega_{ij} = \omega_{ij} + \varepsilon\ \omega^*_{ij} \quad .$$

Then, for a dual spherical motion, we may write formulas which are analogous to those for a real spherical motion [2]. A dual spherical motion of D^3_n relative to $D_f{}^3$, denoted by $D^3{}_n/D_f{}^3$, is generated by the transformation

$$X_n = A_f X_f$$

where X_n and X_f are 3×1 dual matrices which correspond to the position vectors of the same dual point with respect to the orthonormal coordinate systems of D^3_n and D_f^3 respectively. A_f is a proper orthogonal 3×3 dual matrix such that if the transpose of A_f is denoted by A_f^T and the inverse by A_f^{-1} then

$$A_f A_f^T = A_f A_f^{-1} = E \quad . \tag{5}$$

Thus, if we denote the column matrices R_n and R_f by

$$R_n = \begin{bmatrix} \vec{R}_{n1} \\ \vec{R}_{n2} \\ \vec{R}_{n3} \end{bmatrix} , \quad R_f = \begin{bmatrix} \vec{R}_{f_1} \\ \vec{R}_{f_2} \\ \vec{R}_{f_3} \end{bmatrix}$$

then the dual motion D^3_n / D^3_f may be expressed by

$$R_n = A_f R_f \qquad \text{or} \qquad R_f = A_f^{-1} R_n \quad . \tag{6}$$

The elements of A_f will be regarded as differentiable functions of the dual single parameter $\tau = t + \varepsilon\, t^*$ for line-systems and of $\tau = t + \varepsilon\, 0$ for ruled surfaces. We will write $A_f = A_f(\tau)$ to indicate that we restrict the discussion to one-parameter spatial motions in 3-dimensional line-space. In order to use the terminology of kinematic the dual parameter τ will be called "time". The value of τ are ordered [6] as follows:

$$\tau_2 = t_2 + \varepsilon t_2^* > \tau_1 = t_1 + \varepsilon t_1^* \rightleftarrows \begin{cases} t_2 > t_1 & \text{in case } t_2 \neq t_1 \\ t_2^* > t_1^* & \text{in case } t_2 = t_1. \end{cases}$$

III. The Pfaffian Vector-Instantaneous Helicoidal Axis.

Equation (5), by differentiation with respect to the dual parameter τ, yields

$$dA_f\, A_f^T + A_f\, dA_f^T = 0 \tag{7}$$

which shows that the dual matrix

$$\Phi_f = dA_f\, A_f^T = dA_f\, A_f^{-1}, \quad \Phi_f = \varphi_f + \varepsilon \varphi_f; \;\; \forall f \tag{8}$$

is skew-symmetric, so we may write

$$\Phi_f = \begin{bmatrix} 0 & \Omega_{f3} & -\Omega_{f2} \\ -\Omega_{f3} & 0 & \Omega_{f1} \\ \Omega_{f2} & -\Omega_{f1} & 0 \end{bmatrix} . \tag{9}$$

Differentiation of the first equation of (6), supposing that $D^3{}_n$ is in instantaneous motion relative to $D_f{}^3$, furnishes

$$d_f R_n = \Phi_f R_n \tag{10}$$

where the subscript f in d_f means that the motion is to be relative to the f^{th}-sphere $D^3{}_f$.

The position vector $\vec{X}_n$ of a dual point $\vec{X}_n$ on $D^3{}_n$ represents a line-system (in the particular case of $\tau = t + \varepsilon\ 0$ the line-system reduces to a regulus) in space S_n and may be expressed as

$$\left.\begin{array}{l} \vec{X}_n = \vec{X}_n{}^T R_n,\ \ X_n{}^T = [X_{n1}, X_{n2}, X_{n3}],\ \ X_{nj} = x_{nj} + \varepsilon x^*{}_{nj} \\ \vec{X}^2{}_n = \sum\limits_{j=1}^{3} X^2{}_{nj} = 1 \end{array}\right\} \tag{11}$$

and its differential velocity, with respect to the unit dual sphere $D^3{}_f$ which represents the space S_f, is

$$d_f\vec{X}_n = d_n X_n{}^T R_n + X_n{}^T d_f R_n \tag{12}$$

or according to (10)

$$d_f\vec{X}_n = (d_n X_n{}^T + X_n{}^T \Phi_f) R_n \quad . \tag{13}$$

If the point $\vec{X}_n$ is fixed on $D^3{}_n$, then its relative differential velocity (the velocity with respect to $D^3{}_n$) vanishes,

$$d_f X_f\ R_n = 0,$$

and (13) becomes

$$d_f\vec{X}_n = X_n{}^T \Phi_f R_n. \tag{14}$$

Defining a new dual vector by

$$\vec{\Omega}_f = \vec{\Omega}_f(\tau) = \vec{\Omega}_f\,(t) + \varepsilon\ t^*\ \dot{\vec{\Omega}}_f \quad \text{or} \quad \vec{\Omega}_f = \vec{\omega}_f + \varepsilon\ \vec{\omega}_f{}^*$$

whose components Ω_{f_1}, Ω_{f_2}, Ω_{f_3} are, in general, the nonzero elements of the matrix Φ_f, Equation (14) may be written as

$$d_f\vec{X}_n = \vec{\Omega}_f \times \vec{X}_n \tag{15}$$

where the cross denotes the vector product and $\vec{\Omega}_f$ is the *instantaneous Pfaffian (differential) dual vector* of the motion $D^3{}_n/D^3{}_f$, closely analogous to the real spherical case |[2], pp. 6|. At any given instant $\tau = t + \varepsilon\ t^*$ of the motion, $\vec{\Omega}_f = \vec{\Omega}_f(\tau)$ is the analogue of the *Darboux Vector* in the differential geometry of space curves [7], and the direction of $\vec{\Omega}_f(\tau)$ passes through the *dual spherical poles* (instantaneous dual spherical centros of rotation) P_n and P_f on $D^3{}_n$ and $D_f{}^3$, respectively. Thus we have

$$\vec{\Omega}_f(\tau) = \Omega_f(\tau)\ \vec{P}_f(\tau) \quad \text{or} \quad \vec{\Omega}_f(\tau) = \Omega_f(\tau)\ \vec{P}_n(\tau)$$

where $\Omega_f(\tau) = |\ \vec{\Omega}_f(\tau)\,| = \Omega_f(t) + \varepsilon\ t^*\ \dot{\Omega}_f\ (t)$ and $\vec{\Omega}_f(\tau)$ is the *instantaneous dual angular differential velocity* of the motion $D^3{}_n/D_f{}^3$, and its real part $\omega_f(\tau)$ and dual part $\omega_f{}^*(\tau)$ correspond to the *pure rotaion* and the *pure translation* of the motion, respectively. In S_n, a unit dual vector $\vec{X}_f(\tau) = \vec{X}_f(t) + \varepsilon\ t^*\ \vec{X}_f(t)$ corresponds to a line-system and the unit dual vector $\vec{P}_f(\tau)$ corresponds to a line which is the axis of the line-system, at instant τ. This line $\vec{P}_f(\tau)$ is called the *instantaneous helicoidal axis* of the spatial motion S_n/S_f which corresponds to the dual motion $D^3{}_n/D^3{}_f$.

Thus the real and dual parts of $d_fR_n = \Phi_f(\tau)\ R_n$ represent the *pure differential rotational velocity* and the *pure differential translational velocity* of the motion S_n/S_f, respectively. These two parts, from (8) and (10) are

$$d_f r_n = \varphi_f(\tau)\ r_n, \qquad d_f r_n^* = \varphi_f(\tau)\ r_n^* + \varphi_f^*(\tau)\ r_n.$$

This separation is based on the following property of the theory of groups [3].

Theorem 2. The 6-parameter group of motions is the commutative product of the 3-parameter group of rotations and the 3-parameter group of translations.

Hence a system of n unit dual spheres represents a system of n reference frames, as one sees in Figure-1 where the reference frame of S_n is performing combined translation and rotation relative to the reference frame of S_{n-1}, which in turn is performing combined motion relative to reference frame S_{n-2}, etc., and finally the reference frame of S_f is performing combined motion relative to reference frame S_1.

IV. Higher-Order Differentials for Systems with Multiple Relative Motion.

The dual matrix equation for the first-order differential of a dual point $\vec{X}_n$ on $D^3{}_n$ which is moving relative to $D^3{}_f$, according to (13), is

$$d_f X_n = d_n X_n - \Phi_f X_n \tag{16}$$

and the second-order differential is

$$d_f^2\, X_n = d^2_n\, X_n - (\Phi_f X_n)^{(1)} \tag{17}$$

where $(\Phi_f X_n)^{(1)}$ is the differential of $(\Phi_f\, X_n)$ with respect to τ. The Equations (16) and (17) can be generalized for k^{th}-order differential as:

$$d_f^k\, X_n = d_n^k\, X_n - (\Phi_f X_n)^{(k-1)} \tag{18}$$

where for Φ_f, from (9), we have

$$\Phi^2{}_f = \begin{bmatrix} -\Omega^2{}_f + \Omega^2{}_{f_1} & \Omega_{f_1}\Omega_{f_2} & \Omega_{f_1}\Omega_{f_3} \\ \Omega_{f_1}\Omega_{f_2} & -\Omega_f{}^2 + \Omega^2{}_{f_2} & \Omega_{f_2}\Omega_{f_3} \\ \Omega_{f_1}\Omega_{f_3} & \Omega_{f_2}\Omega_{f_3} & -\Omega^2{}_f + \Omega^2{}_{f_3} \end{bmatrix};$$

and

$$\Phi^3_f = -\,\Omega^2_f \Phi_f, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{19}$$

and therefore, for the higher degrees of Φ_f and its higher-order differentials $\Phi^{(k)}{}_f$ we may write the recursion formulas

$$\left.\begin{array}{l} \Phi_f{}^{2m+1} = (-1)^m\, \Omega_f{}^{2m}\, \Phi_f; \quad \Phi_f{}^{2m+2} = (-1)^m\, \Omega_f{}^{2m}\, \Phi_f{}^2 \\ \dot{\Phi}_f{}^{2m+1} = (-1)^m\, \dot{\Omega}_f{}^{2m}\, \dot{\Phi}_f; \quad \dot{\Phi}_f{}^{2m+2} = (-1)^m\, \dot{\Omega}_f{}^{2m}\, \dot{\Phi}_f{}^2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \overset{(k)}{\Phi}_f{}^{2m+1} = (-1)^m\, \overset{(k)}{\Omega}_f{}^{2m}\, \overset{(k)}{\Phi}_f; \quad \overset{(k)}{\Phi}_f{}^{2m+2} = (-1)^m\, \overset{(k)}{\Omega}_f{}^{2m}\, \overset{(k)}{\Phi}{}^2_f \end{array}\right\} \tag{20}$$

where

$$\dot{\Phi}_f = d\Phi_f, \quad \overset{(k)}{\Phi}_f = d^k\Phi_f \quad \text{and} \quad \Omega_f = |\vec{\Omega}_f|\,;$$

$$\dot{\Omega}_f = d\Omega_f, \quad \overset{(k)}{\Omega}_f = d^k\,\Omega_f$$

are differential with respect to the dual variable $\tau = t + \varepsilon\, t^*$.

The angular and translational differential velocities of S_n relative to S_{n-1} are specified by the vectors $\vec{\omega}_n(\tau)$ and $\vec{\omega}_n{}^*(\tau)$ which are the real and dual parts of the instantaneous pfaffian vector $\vec{\Omega}_n(\tau)$, the angular and translational differential velocities of S_{n-1} relative to S_{n-2} by the vectors $\vec{\omega}_{n-1}(\tau)$ and $\vec{\omega}^*{}_{n-1}(\tau)$, etc. The dual position vector of a point $\vec{X}_n$ on $D^3{}_n$, representing the line space S_n, may be expressed relative to $D^3{}_f$, representing the line space S_f, as follows:

$$\vec{X}_n = X^T{}_n\, A_f R_f \quad . \tag{21}$$

If S_f is considered as the first body S_1 and S_n as the n^{th} body in the system, then there are (n—2) more bodies to pass from S_n to S_1 and we may write

$$A_f = A_2 A_3 \ldots A_n \quad . \tag{22}$$

Hence from Equation (6)

$$R_n = A_2 A_3 \ldots A_{n-1}\, A_n R_f \tag{23}$$

and (21) reduces to

$$\vec{X}_n = X^T{}_n \; A_2 \; A_3 \ldots A_{n-1} \; A_n \; R_f \tag{24}$$

so that the displacement of S_n with respect to S_1 is given by the dual differential (taking $f = 1$)

$$d_1\vec{X}_n = d_n X_n{}^T \; R_n + X_n{}^T \; [dA_2 A_3 \ldots A_{n-1} \; A_n + A_2 \; dA_3 \; A_4 \ldots A_{n-1} \; A_n + \ldots + A_2 \; A_3 \ldots dA_n] \; R_1 .$$

From (23), since we have

$$R_1 = A_n{}^T \, A^T{}_{n-1} \ldots A_2{}^T \, R_n \, ,$$

we obtain

$$d_1\vec{X}_n = d_n \, X_n{}^T \, R_n + X_n{}^T (dA_2 \; A_2{}^T) + A_2 \; (dA_3 \; A_3{}^T) \; A_2{}^T + \ldots + A_2 \; A_3 \ldots (dA_n \; A_n{}^T) \ldots A_3{}^T \; A_2{}^T] \; R_n$$

or according to (8)

$$d_1\vec{X}_n = [d_n X_n{}^T + X_n{}^T \, [(\Phi_2 + A_2 \; \Phi_3 \; A_2{}^T + A_2 \; A_3 \; \Phi_4 A_3{}^T \; A_2{}^T + \ldots + A_2 \; A_3 \ldots \Phi_n \ldots A_3{}^T \; A_2{}^T)] \; R_n \, .$$

Since $\Phi_i (i=1,2,3,\ldots)$ are skew-symmetric matrices we eventually have

$$d_1 X_n = d_n X_n - [\Phi_2 + A_2{}^T \, \Phi_3 A_2 + A_2{}^T \, A_3{}^T \, \Phi_4 \, A_3 \, A_2 + \ldots + A_2{}^T \; A_3{}^T \ldots \Phi_n \ldots A_3 \; A_2] \; X_n \, . \tag{25}$$

Comparing (16) and (25) we obtain

$$\Phi_1 = \Phi_2 + A_2{}^T \; \Phi_3 \; A_2 + (A_3 \, A_2)^T \; \Phi_4 \; (A_3 \; A_2) + \ldots + (A_{n-1} \ldots A_3 \; A_2)^T \; \Phi_n (A_{n-1} \ldots A_3 \; A_2)$$

and if we denote the similar matrices of skew-symmetric matrices Φ_i by H_i, respectively, the last equation reduces to

$$\Phi_1 = \Phi_2 + H_3 + H_4 + \ldots + H_n \tag{26}$$

where H_i also are 3×3 skew-symmetric matrices.

Hence (25) becomes

$$d_1 X_n = d_n X_n - [\Phi_2 + H_3 + \ldots + H_n] \; X_n \, . \tag{27}$$

The k^{th}-order differential of a point X_n on $D^3{}_n$, moving relative to sphere $D^3{}_1$, is

$$d_1{}^k X_n = d_n{}^k X_n - \{[\Phi_2 + H_3 + \ldots + H_n] \; X_n\}^{(k-1)} \tag{28}$$

where for the skew-symmetric matrices H_i there exist recursion formulas similar to those for Φ_i in (20).

The real and dual parts of (28) are equivalent to the expressions of Sherby and Chmielewski [4] for the k^{th}-order differential angular and translational velocities of a moving reference frame as seen by an observer in any arbitrary reference frame in the system with multiple relative motion.

V. Geometrical Interpretations of Differential Expressions for a System of n Moving Reference Frames.

In Section III we see that the matrix Φ_f in Equation (14) corresponds to the instantaneous pfaffian dual vector $\vec{\Omega}_f$ in Equation (15). Similarly, matrices $\Phi_1, \Phi_2, \ldots, \Phi_n$ correspond to pfaffian vectors $\vec{\Omega}_1, \vec{\Omega}_2, \ldots, \vec{\Omega}_n$, respectively. Thus these similar matrices $H_3, H_4, \ldots, H_n$ correspond to the transposed pfaffian vectors $\vec{\Psi}_3, \vec{\Psi}_4, \ldots, \vec{\Psi}_n$, respectively. If we premultiply (26) by X_n^T and postmultiply by R_n, then according to (14) and (15) the result is

$$\vec{\Omega}_1(\tau) = \vec{\Omega}_2(\tau) + \vec{\Psi}_3(\tau) + \ldots + \vec{\Psi}_n(\tau) \tag{29}$$

where $\vec{\Omega}_i = \vec{\omega}_i + \varepsilon \vec{\omega}_i^*$ and $\vec{\Psi}_i = \vec{\psi}_i + \varepsilon \vec{\psi}_i^*$.

In particular, for a kinematical system with three relative motions, Equation (29) furnishes

$$\vec{\Omega}_1 = \vec{\Omega}_2 + \vec{\Psi}_3 \ . \tag{30}$$

In the special case of $\tau = t + \varepsilon\ 0$, (30) is the same combination which is given in the book by Müller [3]. In order to see the geometrical interpretaion of (30) we write it in the form

$$\Psi_3\ \vec{P}_3 = \Omega_1\ \vec{P}_1 - \Omega_2\ \vec{P}_2, \qquad \vec{P}_1^{\,2} = \vec{P}_2^{\,2} = \vec{P}_3^{\,2} = 1$$

and define a new unit dual vector $\vec{N} = \vec{N}(\tau)$ by

$$\vec{N} = \vec{P}_1 \times \vec{P}_2 \ . \tag{31}$$

Then, because of $\vec{N}.\vec{P}_1 = \vec{N}.\vec{P}_2 = \vec{N}.\vec{P}_3 = 0$, any line of the corresponding line-system $\vec{N}(\tau) = \vec{N}(t) + \varepsilon\ t^* \ \dot{\vec{N}}(t)$ orthogonally intersects each line of the line (axis)-Systems $\vec{P}_1(\tau)$, $\vec{P}_2(\tau)$ and $\vec{P}_3(\tau)$. Hence we have the following theorem:

Theorem 3. In general motion of 3-dimensional Euclidean spaces S_1, S_2, S_3, with respect to each other, their three instantaneous helicoidal axis-systems have a common perpendicular line-system.

In other words, all lines of these three (axis) line-systems belong to a *normal net*, i.e. they belong to the set of ∞^2 straight lines which orthogonally intersect the straight line $\dot{N}(\tau)$. This net is a *linear congruence* whose directrices are the lines $\vec{N}(\tau)$ and $\vec{N}_\infty(\tau)$, where $\vec{N}_\infty(\tau)$ is the *infinite-line* of the plane whose normal is $\vec{N}(\tau)$.

On the other hand, in 3-dimensional Euclidean space there is a one-to-one correspondence between the instantaneous helicoidal motions and the *linear complexes* whose axes and *pitches* are the axes and pitches of the corresponding instaneous helicoidal motions {[3], pp. 245–247}. All lines of any complex are normals of the *helicoidal orbit* of a point of moving space, and vice versa. Therefore in moving space S_3 if we consider all line-systems $\vec{X}(\tau)=\vec{x}+\varepsilon\vec{x}^*$ which orthogonally intersect each one of the axes $\vec{P}_1(\tau)$, $\vec{P}_2(\tau)$ and $\vec{P}_3(\tau)$, then there are three linear complexes at each instant τ

$$\left.\begin{array}{l} \vec{\omega}_1.\vec{x}^* + \vec{\omega}_1^*.\vec{x} = 0 \\ \vec{\omega}_2.\vec{x}^* + \vec{\omega}_2^*.\vec{x} = 0 \\ \vec{\Psi}_3.\vec{x}^* + \vec{\Psi}_3^*.\vec{x} = 0\ , \end{array}\right\} \qquad (32)$$

which correspond to $\vec{\Omega}_1(\tau)$, $\vec{\Omega}_2(\tau)$, $\vec{\Psi}_3(\tau)$, respectively. Joining (30) and (32) we have

$$\vec{\psi}_3 . \vec{x}^* + \vec{\psi}_3^* . \vec{x} = (\vec{\omega}_1 . \vec{x}^* + \vec{\omega}_1^* . \vec{x}) - (\vec{\omega}_2 . \vec{x}^* + \vec{\omega}_2^* . \vec{x}) = 0 \quad (33)$$

which shows that these three linear complexes form a *pencil of linear complexes.*

If we restrict the dual variable, time τ, to

$$\tau \equiv \tau(u) = t(u) + \varepsilon t^*(u), \quad \left(\frac{dt(u)}{du} \neq 0\right)$$

then u can be considered as an ordinary time-variable with respect to which an ordinary motion is determined. In this special case the previous results are also valid. For example in the book by Müller [3] we see that these results hold in the special case of $\tau = t + \varepsilon\, 0$ and $n = 3$.

Now we can generalize these results to kinematical systems with n elements. In general, if we have n rigid bodies in motion relative to each other, and these motions depend on a dual parameter τ, then at each instant τ we have $\binom{n}{2}$ relative instantaneous helicoidal axes. Any three of these axes have a common perpendicular line $\vec{N}(\tau)$, and the number of these lines $\vec{N}(\tau)$ is $\binom{n}{3}$ at that instant τ. The set of $\binom{n}{2}$ axes and corresponding $\binom{n}{3}$ lines $\vec{N}(\tau)$ comprise a *configuration* which is the spatial generalization of the configuration of pole points in one-real parameter planar kinematical systems with multiple relative motion [[1], pp. 22]. The line $\vec{N}(\tau)$ corresponds to the pole line (polgerade) of the planar case.

In the general motion of the system, there are $\binom{n}{2}$ linear complexes of lines of the space S_n; n successive complexes correspond to $\vec{\Omega}_1(\tau)$, $\vec{\Omega}_2(\tau)$, $\vec{\Psi}_3(\tau)$,..., $\vec{\Psi}_n(\tau)$, respectively, and their equations are

$$\left.\begin{array}{l} \vec{\omega}_1 . \vec{x}^* + \vec{\omega}_1^* . \vec{x} = 0 \\ \vec{\omega}_2 . \vec{x}^* + \vec{\omega}_2^* . \vec{x} = 0 \\ \vec{\psi}_3 . \vec{x}^* + \vec{\psi}_3^* . \vec{x} = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \vec{\psi}_n . \vec{x}^* + \vec{\psi}_n^* . \vec{x} = 0 \end{array}\right\} . \qquad (34)$$

In order to find the relation involving these linear complexes we write the real and dual parts of $\vec{\Omega}_1(\tau)$ in (29) and we have

$$\vec{\omega}_1 = \vec{\omega}_2 + \vec{\psi}_3 + \ldots + \vec{\psi}_n \ ,$$

$$\vec{\omega}_1^* = \vec{\omega}_2^* + \vec{\psi}_3^* + \ldots + \vec{\psi}_n^* \ .$$

Then we substitute these two parts into the first equation of (34) and obtain

$$\vec{\omega}_1 . \vec{x}^* + \vec{\omega}_1^* . \vec{x} = (\vec{\omega}_2 . \vec{x}^* + \vec{\omega}_2^* . \vec{x}) + (\vec{\psi}_3 . \vec{x}^* + \vec{\psi}_3^* . \vec{x}) + \ldots$$
$$+ (\vec{\psi}_n . \vec{x}^* + \vec{\psi}_n^* . \vec{x}) = 0 \qquad (35)$$

which is the generalization of (33) to a kinematical system of n reference frames. Hence we have the following theorem as a result.

Theorem 4. A kinematical system of n reference frames in one-dual parameter motion relative to each other has $\binom{n}{2}$ helicoidal axis (line)-systems and $\binom{n}{2}$ corresponding linear complexes. n successive linear complexes which correspond to $\vec{\Omega}_1(\tau)$, $\vec{\Omega}_2(\tau)$, $\vec{\Psi}_3(\tau)$,..., $\vec{\Psi}_n(\tau)$ form a pencil of linear complexes.

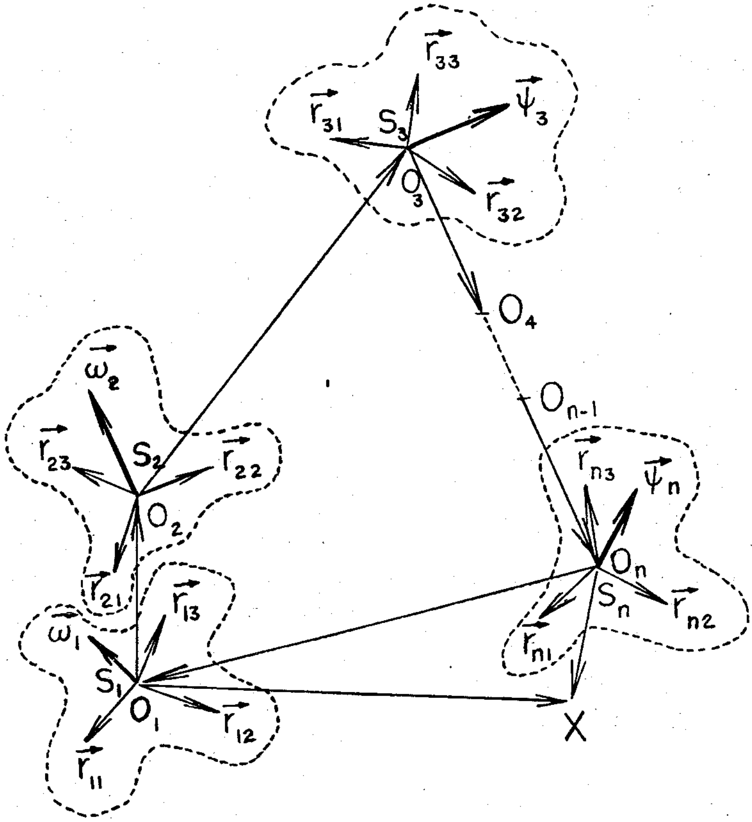
r33
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Figure 1

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