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Acceleration Axes in Spatial Kinamatics II.

by

H. HİLMİ HACISALİHOĞLU

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H. HİLMİ HACISALİHOĞLU*

Ankara Fen Fakültesi

ABSTRACT

In this paper we derived the geometric properties of three acceleration axes in R^3. The velocity and acceleration distributions corresponding to the axes are derived. Finally we discuss the special cases.

I. INTRODUCTION

Acceleration axes in spherical kinematics are discussed in the paper of Bottema [1]. In spatial kinematics, the location and reality are derived in one of the author's papers [2]. This paper is a continuation of [2]. In section II geometric properties of these axes, the velocity and acceleration distributions corresponding to the axes are derived.

In section III, we discuss the special cases.

For the basic concepts and all of our notations we refer the paper [2].

II. CONFIGURATION OF THE ACCELERATION AXES

The three acceleration axes l_i are determined by the twelve Plücker line coordinates of

$$\vec{W} = \frac{\vec{\Psi}}{\Psi} = \frac{\vec{\psi} + \varepsilon \vec{\psi}^*}{\Psi} \text{ and } \vec{F} = \frac{\dot{\vec{\Psi}}}{\dot{\Psi}} = \frac{\dot{\vec{\psi}} + \varepsilon \dot{\vec{\psi}}^*}{\dot{\Psi}}. \quad (2\text{-}1)$$

* Cebir-Geometri Kürsüsü, Ankara Üniversitesi Fen Fakültesi.

Since these coordinates must verify the relations

$$\vec{W}^2 = 1, \quad \vec{F}^2 = 1$$

and the real and dual parts of {(3-19), [2]} the number of configurations of these axes is at most ∞^6. A configuration of axes contains in general three skew lines. Now we shall try to obtain some properties of any configuration of three skew acceleration axes.

Leaving for section III the special case in which an acceleration axis is orthogonal to the instantaneous axis of rotation, we suppose that the three axes are real and distinct, and we define the orientation of the line l_i by the condition that its positive direction shall make a dual angle

$$\Theta_i = \theta_i + \varepsilon\theta^*_i$$

with $\vec{W}$, such that θ_i is an acute angle. With this orientation we define the unit vector $\vec{V}_i$ which corresponds to l_i, the endpoint being S_i. We denote the dual angle between the dual vectors $\vec{V}_i$ and $\vec{V}_j$ by

$$\varnothing_{ij} = \varphi_{ij} + \varepsilon\varphi^*_{ij}$$

which, therefore, denote the dual length of the side S_iS_j of the dual spherical triangle $S_1S_2S_3$.

Since the vectors $\vec{V}_i$ satisfy {(3-19), [2]} we have

$$\Lambda_i\Psi'^2 \vec{V}_i - (\vec{\Psi}' . \vec{V}_i) \ \vec{\Psi}' - \dot{\vec{\Psi}}' x \vec{V}_i = 0, \ (i=1,2,3). \qquad (2\text{-}2)$$

Taking the scalar product of the left-hand-side and $\vec{V}_j$, the result is

$$\Lambda_i \Psi'^2 \cos\varnothing_{ij} - \Psi'^2 \cos\Theta_i \cos\Theta_j - \dot{\vec{\Psi}}' . (\vec{V}_i \ x \ \vec{V}_j) = 0 \quad (2\text{-}3)$$

and adding this to the analogous equation with i and j interchanged we obtain

$$(\Lambda_i + \Lambda_j) \cos\varnothing_{ij} = 2 \cos \Theta_i \cos\Theta_j \tag{2-4}$$

or as real and dual parts

$$\left.\begin{aligned}(\lambda_i + \lambda_j) \cos \varphi_{ij} &= 2 \cos\theta_i \cos\theta_j \\ \varphi^*_{ij} &= \frac{(\lambda^*_i+\lambda^*_j) \cos\varphi_{ij}+2\,[\theta^*_j \cos\theta_i \sin\theta_j+\theta^*_i\cos\theta_j\sin\theta_i]}{2 \cot\varphi_{ij} \cos\theta_i \cos\theta_j}\end{aligned}\right\} \tag{2-5}$$

In the case of $i = j$, since $\varphi_{ij} = \varphi^*_{ij} = 0$, (2-4) gives

$$\Lambda_i = \cos^2 \Theta_i \tag{2-6}$$

and therefore we have

$$\lambda_i = \cos^2\theta_i,\ \lambda^*_i = - \theta^*_i \sin 2\theta_i \text{ or } \theta^*_i = - \frac{\lambda^*_i}{\sin2\theta_i}. \tag{2-7}$$

On the other hand from {(3-21), [2]} we may write

$$\left.\begin{aligned}\Lambda_1 + \Lambda_2 + \Lambda_3 &= 1 \\ \Lambda_1\Lambda_2 + \Lambda_1\Lambda_3 + \Lambda_2\Lambda_3 &= K \\ \Lambda_1\Lambda_2\Lambda_3 &= K \cos^2\nabla\end{aligned}\right\} \tag{2-8}$$

where the first equation gives a simple geometrical meaning of the dual roots of {(3-21), [2]} which yields the relation

$$\left.\begin{aligned}&\sum_{i=1}^{3} \cos^2 \Theta_i = 1 \\ \text{or } &\sum_{i=1}^{3} \cos^2\theta_i = 1 \quad \text{and} \quad \sum_{i=1}^{3} \theta^*_i \sin 2\theta_i = 0\end{aligned}\right\} \tag{2-9}$$

for the angles and distances of $\vec{W}$ and the acceleration axes. The second equation and first equation of (2-8) give another expression of k^* as follows

$$\sum_{i=1}^{3} \lambda_i \lambda_i^* = - k^* . \tag{2-10}$$

Comparing (2-10) and {(3-22), [2]} we may write

$$\frac{\psi^*}{\psi} = \frac{1}{2}\frac{\dot{\psi}^*}{\dot{\psi}} + \frac{1}{4k}\sum_{i=1}^{3} \lambda_i \lambda^*_i \qquad (2\text{-}11)$$

where $\frac{\psi^*}{\psi}$ is the pitch of the instantaneous helicoidal motion.

Hence if $\sum_{i=1}^{3} \lambda_i \lambda^*_i = 0$ i. e. in the cases b) and c) given before, the pitch of the instantaneous motion whose axis is $\vec{\Psi}$ equals the half of the pitch of the instantaneous motion whose axis is $\dot{\vec{\Psi}}$. Also we may write

$$\lambda_i \lambda^*_i = 2\theta^*_i \sin\theta_i \cos^3\theta_i .$$

Eventually the third equation of (2-8) gives the relation

$$\frac{k^*}{k} = 2\alpha^* \operatorname{tg}\alpha + \sum_{i=1}^{3} \frac{\lambda^*_i}{\lambda_i} \qquad (2\text{-}12)$$

where

$$\frac{\lambda^*_i}{\lambda_i} = -2\,\theta^*_i \operatorname{tg}\theta_i .$$

Hence from {(3-22), [2]}, we have another expression

$$\frac{\psi^*}{\psi} = \frac{1}{2}\left[\frac{\dot{\psi}^*}{\dot{\psi}} - \alpha^* \operatorname{tg}\alpha + \sum_{i=1}^{3} \theta^*_i \operatorname{tg}\theta_i\right] \qquad (2\text{-}13)$$

for the pitch of the instantaneous helicoidal motion.

For the sake of brevity we write

$$\Lambda_i^{\frac{1}{2}} = \cos\Theta_i = U_i = u_i + \varepsilon u^*_i .$$

Then it follows from (2-4) that

$$\cos \varnothing_{ij} = \frac{2U_i\ U_j}{U_i^2+U_j^2} \tag{2-14}$$

and, therefore, we have

$$\cos\varphi_{ij} = \frac{2u_iu_j}{u_i^2+u_j^2},\ \varphi^*_{ij} = \frac{u_i^*u_j - u_iu_j^*}{u_i^2+u_j^2},\ \sin\varphi_{ij} = \frac{u_i^2 - u_j^2}{u_i^2+u_j^2}. \tag{2-15}$$

Since $U_i \neq U_j$ and θ_i is an acute angle, φ_{ij} also is an acute angle i. e. $0 < \cos\varphi_{ij} < 1$.

From (2-15) $\varphi^*_{ij} = 0$ implies that

$$\frac{\theta_i^*}{\theta_j^*} = \frac{tg\theta_j}{tg\theta_i}$$

and $\varphi_{ij} = 0$ implies that

$$\theta_i = \theta_j.$$

Hence we conclude the following theorems:

Theorem 2-1. In the spatial motion H/H′ the necessary and sufficient condition for the intersection of any two acceleration axes l_i and l_j is that their minimal distances and slopes with respect to the inistantaneous axis $\vec{W}$ of the motion have an inverse ratio.

Theorem 2-2. In H/H′ two acceleration axes l_i and l_j are parallel if and only if their angles with the instantaneous axis are equal.

In terms of U_i there are similar expressions for the dual angles of the dual spherical triangle $S_1S_2S_3$, which ve denote by

$$\Delta_i = \delta_i + \varepsilon\delta_i^*.$$

Then $\pi - \Delta_i$ is the dual angle between common perpendiculars of $(\vec{V}_i, \vec{V}_j)$ and $(\vec{V}_i, \vec{V}_k)$.

According to the cosine rule for a spherical triangle we may write

$$\cos\Delta_i = \frac{\cos\varnothing_{jk} - \cos\varnothing_{ij}\ \cos\varnothing_{ik}}{\sin\varnothing_{ij}\ \sin\varnothing_{ik}} \tag{2-16}$$

Where $\sin \varnothing_{ij}$, from (2-14), is

$$\sin^2 \varnothing_{ij} = \frac{U_i^2 - U_j^2}{U_i^2 + U_j^2} \tag{2-17}$$

or according to (2-14) and (2-17)

$$\cos^2\Delta_i = \cos^2 \varnothing_{jk} \tag{2-18}$$

and the real and dual parts of the last equality are

$$\cos^2 \delta_i = \cos^2\varphi_{jk}, \tag{2-19}$$

$$\delta^*_i \operatorname{Sin} 2\delta_i = \varphi^*_{jk} \operatorname{Sin} 2\varphi_{jk}. \tag{2-20}$$

Then (2-19) implies that

$$\delta_i = \varphi_{jk} \text{ or } \delta_i + \varphi_{jk} = \pi . \tag{2-21}$$

(2-20) and (2-21) give us

$$\delta^*_i = \varphi^*_{jk}. \tag{2-22}$$

Thus we have the following theorem in conclusion:

Theorem 2-3. For three skew acceleration axes there are three skew lines such that each of them is a common perpendicular between a pair of axes. The angle between two of these common perpendiculars is equal to, or the supplement of, the angle between the two acceleration axes which have the third common perpendicular. The minimal distance of two common perpendiculars is equal to the minimal distance of two acceleration axes which have the third common perpendicular.

From (2-14) and (2-18) we conclude that the triangle $S_1 S_2 S_3$ is neither right angled nor isosceles, but two of its angles are equal to, and the third is the supplement of, the opposite side. This is also a consequence of Delambre's analogy in the triangle $S_1S_2S_3$ [1]. Hence there are the following theorems:

Theorem 2-4. The angle and minimal distance between any two of three acceleration axes are different from the angle and minimal distance of each other pair.

Theorem 2-5. Any two of three common perpendiculars to two of three acceleration axes can not be orthogonal.

From (2-17) we can write the following relation for the dual angles $\varnothing_{ij}$;

$$\frac{1+\sin\varnothing_{23}}{1-\sin\varnothing_{23}}\;\frac{1-\sin\varnothing_{31}}{1+\sin\varnothing_{31}}\;\frac{1+\sin\varnothing_{12}}{1-\sin\varnothing_{12}}=1 \quad (2\text{-}23)$$

or

$$\left.\begin{aligned} &\sin\varphi_{12}+\sin\varphi_{23}-\sin\varphi_{31}=\sin\varphi_{12}\,\text{sn}\varphi_{23}\,\sin\varphi_{31}\\ &\frac{\varphi^*_{12}\cos\varphi_{12}+\varphi^*_{23}\cos\varphi_{23}-\varphi^*_{31}\cos\varphi_{31}}{\varphi^*_{12}\text{cotg}\varphi_{12}+\varphi^*_{23}\text{cotg}\varphi_{23}+\varphi^*_{31}\text{cotg}\varphi_{31}}=\sin\varphi_{12}\sin\varphi_{23}\sin\varphi_{31} \end{aligned}\right\} \quad (2\text{-}24)$$

a) The Position of the Angular Velocity and Angular Accecelleration Vectors $\vec{\Psi}$ *and* $\dot{\vec{\Psi}}$ *with Respect to the Accelaration Axes* $\vec{V}_i$:

In this section at first we shall prove the two following theorems:

Theorem 2-6. The common perpendicular to $\vec{V}_i$ and $\vec{W}$ orthogonally intersects the common perpendicular to $\vec{V}_i$ and $\vec{F}$.

Proof: The acceleration of a point on l_i is

$$\vec{J}_i=-\Psi'^2\ \vec{V}_i+(\vec{\Psi}.\vec{V}_i)\ \vec{\Psi}+\dot{\vec{\Psi}}\times\vec{V}_i. \quad (2\text{-}25)$$

The orthogonal directions to l_i are

$$\dot{\vec{V}}_i=\vec{\Psi}\times\vec{V}_i \quad (2\text{-}26)$$

where $\dot{\vec{V}}_i=d\vec{V}_i$. According to the definition of l_i, the orthogonal component to l_i of $\vec{J}_i$ is zero. Thus

$$\vec{J}_i.\ \dot{\vec{V}}_i=0 \quad (2\text{-}27)$$

or from (2-25) and (2-26)

$$(\dot{\vec{\Psi}} x \vec{V}_i) . (\vec{\Psi} \text{ x } \vec{V}_i) = 0$$

or from (2-1)

$$(\vec{F} \text{ x } \vec{V}_i).(\vec{W} \text{ x } \vec{V}_i) = 0 \quad (2\text{-}28)$$

this completes the proof.

Definition 2-7. In the moving space H, a definite line $\vec{X}$, during the motion H/H′, generates a surface in H′ which we call the *orbit surface* of $\vec{X}$.

Theorem 2-8. At the instant t, the point of striction of the orbit surface of $\vec{V}_i$ is the intersection point of the common perpendiculars to $\vec{W}$, $\vec{V}_i$ and $\vec{F}$, $\vec{V}_i$.

Proof: Let us denote the common perpendicular to $\vec{W}$ and $\vec{V}_i$ by $\vec{Y}$ and the neighboring generators by $\vec{\bar{V}}_i$. Then we have

$$\vec{Y} = \vec{W} \text{ x } \vec{V}_i \quad (2\text{-}29)$$

$$\vec{\bar{V}}_i = \vec{V}_i + \dot{\vec{V}}_i$$

$$\vec{\bar{V}}_i = \vec{V}_i + \Psi (\vec{W} \text{ x } \vec{V}_i) \quad (2\text{-}30)$$

$$\vec{\bar{V}}_i = \vec{V}_i + \Psi \vec{Y}.$$

If we denote the common perpendicular to $\vec{V}_i$ and $\vec{\bar{V}}_i$ by $\vec{Z}$ then

$$\vec{Z} = \vec{V}_i \text{ x } \vec{\bar{V}}_i = \vec{V}_i \text{ x } \vec{Y}. \quad (2\text{-}31)$$

Since the intersection point of $\vec{V}_i$ and $\vec{Z}$ is the striction point of generator $\vec{V}_i$, in order to complete the proof we must show that the two lines $\vec{Y}$ and $\vec{Z}$ meet at a right angle. From (2–31)

$$\vec{Y} \cdot \vec{Z} = \vec{Y} \cdot (\vec{V}_i \times \vec{Y})$$

$$\vec{Y} \cdot \vec{Z} = 0 .$$

On the other hand according to (2-28) the common perpendicular to $\vec{F}$ and $\vec{V}_i$ orthogonally intersects $\vec{Y}$. Hence the lines $\vec{Y}$, $\vec{Z}$ and $\vec{F} \times \vec{V}_i$ meet at a right angle at the striction point of $\vec{V}_i$ (Fig. 2-1).

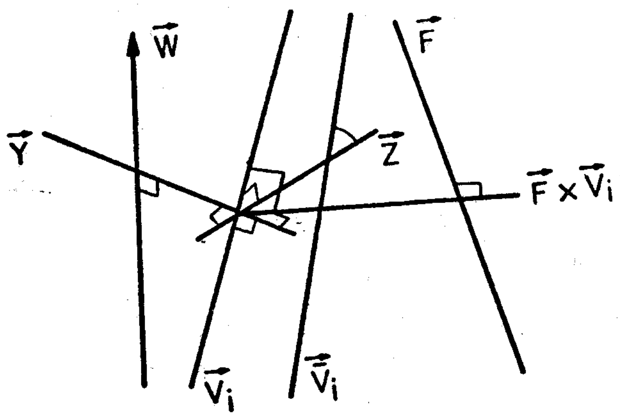

FIG. 2-1

a) Interchange of $\vec{W}$ *and* $\vec{F}$:

If $\Sigma_i = \sigma_i + \varepsilon\sigma^*_i$ is the dual angle between $\vec{V}_i$ and $\vec{F}$ then from (2-28) we obtain

$$\cos \Theta_i \cos \Sigma_i = \cos \nabla . \qquad (2\text{-}32)$$

On the other hand according to Theorem (2-1), if W, E and S_i are the endpoints of unit dual vectors $\vec{W}$, $\vec{F}$ and $\vec{V}_i$ then the pair of W, F is seen from S_i as a right dual angle. Hence Fig. (2-2) illustrates two more unit dual vectors $\vec{B}_k$ and $\vec{B}_j$ which comp-

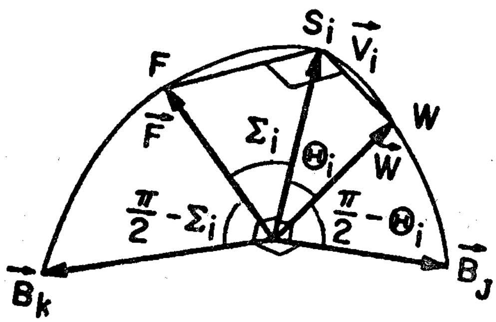

FIG. 2-2

lete $\vec{S}_i$ to an orthonormal dual system. Taking the scalar product of the left-hand side of (2-2) and $\vec{B}_j$ the result is

$$(\vec{\Psi}.\vec{V}_i) \quad (\vec{\Psi}.\vec{B}_j) - (\vec{V}_i \times \dot{\vec{\Psi}}).\vec{B}_j = \Lambda_i \Psi^2 \; (\vec{V}_i . \; \vec{B}_j)$$

or

$$\cos \Theta_i \quad \sin \Theta_i = \frac{\dot{\Psi}}{\Psi^2} \sin \Sigma_i . \qquad (2\text{-}33)$$

If we eliminate Σ_i from (2-32) and (2-33) we again obtain {(3-21), [2]}, with $\Lambda_i = \cos^2 \Theta_i$. And we also obtain, from (2-32) and (2-33), the following relation

$$\cos \Sigma_i \; \sin \Sigma_i = \frac{\Psi^2 \cos \nabla}{\dot{\Psi}} \sin \Theta_i \qquad (2\text{-}34)$$

which is the same form as (2-33) . Hence we may express the following theorem:

Theorem 2-9. During the one-parameter motion H/H′, $\vec{W}$ and $\vec{F}$ may be interchanged, leaving the acceleration axes l_i invariant.

A more precise proof of this theorem may be given in the same way as Bottema [1], treating everything as dual.

b) Position of $\vec{F}$ and $\vec{W}$ with respect to lines l_1, l_2, l_3:

If we subtract from (2-3) the analogous equation with i and j interchanged, the result is

$$\frac{1}{2}\,\Psi'^2\quad(\Lambda_i - \Lambda_j)\quad\cos\varnothing_{ij} = \dot{\vec{\Psi}}.\quad(\vec{V}_i \times \vec{V}_j). \tag{2-35}$$

If we denote the dual spherical distances of W and F from the side S_i S_j by $P_k = p_k + \varepsilon p^*_k$ and $Q_k = q_k + \varepsilon q^*_k$ (Fig. (2-3)) then from (2-35) we have

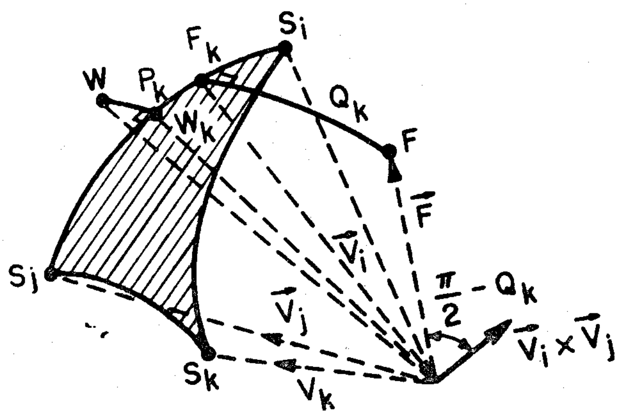

FIG. 2-3

$$\frac{1}{2}\,\Psi'^2\quad(\Lambda_i - \Lambda_j)\quad\cos\varnothing_{ij} = \dot{\Psi}\,\sin\varnothing_{ij}\,\sin Q_k \tag{3-36}$$

or according to (2-14) and (2-17)

$$\sin Q_k = \frac{\Psi'^2}{\dot{\Psi}}\,U_i\;U_j$$

$$\sin Q_k = \frac{\Psi'^2}{\dot{\Psi}}\,\cos\Theta_i\;\cos\Theta_j. \tag{2-37}$$

And from the last equality of (2-8) according to (2-6), (2-37) becomes

$$\sin Q_k = \cos \Sigma_k \tag{2-38}$$

or

$$\left.\begin{aligned} \sigma_k + q_k &= \frac{\pi}{2} \\ q^*_k &= - \sigma^*_k . \end{aligned}\right\} \tag{2-39}$$

There is the analogus formula for W: If we take the scalar product of the right-hand side (2-2) and $\vec{V}_i \times \vec{V}_j$ the result is

$$\cos\Theta_i \sin P_k \sin \varnothing_{ij} + \frac{\dot{\Psi}}{\Psi^2} \cos \nabla \left(\frac{\cos \varnothing_{ij}}{\cos\Theta_i} - \frac{1}{\cos\Theta_j}\right) = 0 \tag{2-40}$$

and from the last equllity of (2-8), (2-6) and (2-17) the equation (2-40) reduces to

$$\sin P_k = \cos\Theta_k \tag{2-41}$$

or

$$\left.\begin{aligned} p_k + \theta_k &= \frac{\pi}{2} \\ p^*_k &= - \theta^*_k . \end{aligned}\right\} \tag{2-42}$$

Hence (2-39) and (2-42) give the following theorem:

Theorem 2-10. The instantaneous rotation axis $\vec{W}$ (or $\vec{F}$) of H/H′ is at equal minimal distance from the common perpendicular to any two acceleration axes and the other acceleration axis. The angle of $\vec{W}$ (or $\vec{F}$) and the common perpendicular and the angle of $\vec{W}$ (or $\vec{F}$) and the axis are complementary angles.

Eventually, from (2-32), (2-38) and (2-41) we may write

$$\sin P_k \; \sin Q_k = \cos \nabla \tag{2-43}$$

or

$$\left.\begin{aligned} \sin p_k \quad \sin q_k &= \cos\alpha \\ p^*_k \operatorname{cotg} p_k + q^*_k \quad \operatorname{cotg} q_k + \alpha^* \quad \operatorname{tg}\alpha &= 0 \end{aligned}\right\} \qquad (2\text{-}44)$$

and from (2-39) and (2-42) the equations (2-44) become

$$\left.\begin{aligned} \cos\sigma_k \quad \cos\theta_k &= \cos\alpha \\ \theta^*_k \operatorname{tg}\theta_k + \sigma^*_k \operatorname{tg}\sigma_k - \alpha^* \operatorname{tg}\alpha &= 0. \end{aligned}\right\} \qquad (2\text{-}45)$$

Replacing (2-45) in (2-13) we obtain another expression for the pitch of instantaneous helicoidal motion H/H′ as follows:

$$\frac{\psi^*}{\psi} = \frac{1}{2}\left[\frac{\dot\psi^*}{\dot\psi} + 2\alpha^* \operatorname{tg}\alpha - \sum_{i=1}^{3} \sigma_i^* \operatorname{tg}\alpha_i\right]. \qquad (2\text{-}46)$$

This is the same form as (2-13); it follows that $\vec{W}$ and $\vec{F}$ may be interchanged, leaving the pitch of instantaneous helicoidal motion H/H′.

b) Some Remarks about Common Perpendiculars to Pairs of the lines $\vec{W}$, $\vec{F}$, $\vec{V}_i$:

Let us define $\vec{L}_i$, $\vec{T}_i$, $\vec{\Gamma}_i$ as follows:

$\vec{L}_i = \vec{V}_j \times \vec{V}_k$ is the common perpendicular to $\vec{V}_j$ and $\vec{V}_k$;

$\vec{T}_i = \vec{W} \times (\vec{V}_j \times \vec{V}_k)$ is the common perpendicular to $\vec{W}$ and $\vec{L}_i$;

$\vec{\Gamma}_i = \vec{F} \times (\vec{V}_j \times \vec{V}_k)$ is the common perpendicular to $\vec{F}$ and $\vec{L}_i$.

Hence $\vec{W}$ is the common perpendicular to three lines $\vec{T}_1$, $\vec{T}_2$, $\vec{T}_3$ and $\vec{F}$ is the common perpendicular to three lines $\vec{\Gamma}_1, \vec{\Gamma}_2, \vec{\Gamma}_3$.

Taking the scalar product of $\vec{T}_i$ and $\vec{\Gamma}_i$ the result is 0. Thus the lines $\vec{T}_i$ and $\vec{\Gamma}_i$ orthogonally intersect each other. Since the common perpendicular to $\vec{T}_i$ and $\vec{\Gamma}_i$ is $\vec{L}_i$ we conclude that the lines $\vec{T}_i$, $\vec{\Gamma}_i$ and $\vec{L}_i$ form a rectangular trihedron. For $i = 1,2,3$ in space H there exist three such trihedrons (Fig. 2-4).

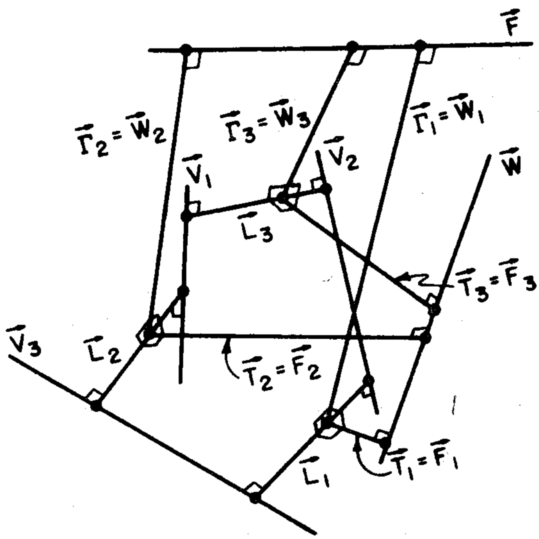

FIG. 2-4

On the other hand if W_k is the projection of W on S_iS_j then

$$(\vec{V}_i \times \vec{V}_j) \cdot \vec{W}_k = 0, \qquad \vec{W}_k \cdot \vec{W} = \cos P_k$$

and since

$$\widehat{WS_i} = \Theta_i, \qquad \widehat{WW_i} = P_i$$

for the dual angle $\widehat{W_1S_2}$ in the dual spherical triangle WS_2W_1 the cosine rule gives us

$$\cos \widehat{W_1S_2} = \frac{\cos\Theta_2}{\cos P_1}$$

or from (2-41)

$$\cos \widehat{W_1S_2} = \frac{\cos\Theta_2}{\sin\Theta_1}$$

and, therefore, we obtain that

$$\cos 2\ \widehat{W_1S_2} = \sin \varnothing_{23}\ . \tag{2-47}$$

The same method for the triangle WW_1S_3 gives us

$$\cos 2\ \widehat{W_1S_3} = -\sin \varnothing_{23}\ . \tag{2-48}$$

Hence we have the following relations for W:

$$\left.\begin{aligned} \widehat{W_1S_2} &= \frac{\pi}{4} - \frac{1}{2}\varnothing_{23}\ ; \quad \widehat{W_1S_3} = \frac{\pi}{4} + \frac{1}{2}\varnothing_{23} \\ \widehat{W_2S_3} &= \frac{\pi}{4} - \frac{1}{2}\varnothing_{31}\ ; \quad \widehat{W_2S_1} = \frac{\pi}{4} + \frac{1}{2}\varnothing_{31} \\ \widehat{W_3S_1} &= \frac{\pi}{4} - \frac{1}{2}\varnothing_{12}\ ; \quad \widehat{W_3S_2} = \frac{\pi}{4} + \frac{1}{2}\varnothing_{12} \end{aligned}\right\} . \tag{2-49}$$

And in the same way if F_k is the projection of F on S_iS_j we have following relations for F:

$$\left.\begin{aligned} \widehat{F_1S_2} &= \frac{\pi}{4} + \frac{1}{2}\varnothing_{23}\ ; \quad \widehat{F_1S_3} = \frac{\pi}{2} - \frac{1}{2}\varnothing_{23} \\ \widehat{F_2S_3} &= \frac{\pi}{4} + \frac{1}{2}\varnothing_{31}\ ; \quad \widehat{F_2S_1} = \frac{\pi}{4} - \frac{1}{2}\varnothing_{31} \\ \widehat{F_3S_1} &= \frac{\pi}{4} + \frac{1}{2}\varnothing_{12}\ ; \quad \widehat{F_3S_2} = \frac{\pi}{4} - \frac{1}{2}\varnothing_{12} \end{aligned}\right\} . \tag{2-50}$$

Comparing (2-49) and (2-50) we conclude that

$$\widehat{W_iS_j} = \widehat{F_iS_k}, \qquad \widehat{W_iS_j} + \widehat{F_iS_j} = \frac{\pi}{2} \tag{2-51}$$

and therefore,

$$\vec{W}_i \cdot \vec{F}_i = 0 \quad . \tag{2-52}$$

Since the common perpendicular to $\vec{W}_i$ and $\vec{F}_i$ is $\vec{L}_i$ we conclude that the three lines $\vec{W}_i$, $\vec{F}_i$ and $\vec{L}_i$ also form a rectangular thrihedron.

Now we are going to show that these two rectanglular trihedorn $\{\vec{L}_i, \vec{T}_i, \vec{\Gamma}_i\}$, $\{\vec{L}_i, \vec{W}_i, \vec{F}_i\}$ are coincident. Since we have

$$\left.\begin{aligned} \xi_1 \vec{W}_1 &= \cos P_1 \vec{W} + \cos \widehat{W_1 S_2}\, \vec{V}_2 + \cos \widehat{W_1 S_3}\, \vec{V}_3 \\ \xi_2 \vec{W}_2 &= \cos P_2 \vec{W} + \cos \widehat{W_2 S_1}\, \vec{V}_1 + \cos \widehat{W_2 S_3}\, \vec{V}_3 \\ \xi_3 \vec{W}_3 &= \cos P_3 \vec{W} + \mathrm{eos} \widehat{W_3 S_1}\, \vec{V}_1 + \cos \widehat{W_3 S_2}\, \vec{V}_2 \end{aligned}\right\} \tag{2-53}$$

where ξ_i $(i=1,2,3)$ are dual coefficients; according to (2-41) we have also

$$\left.\begin{aligned} \xi_1 \vec{W}_1 \times \vec{L}_1 &= \sin\theta_1 \vec{W} \times \vec{L}_1 \Rightarrow \vec{W}_1 \times \vec{L}_1 = \vec{T}_1 \\ \xi_2 \vec{W}_2 \times \vec{L}_2 &= \sin\theta_2 \vec{W} \times \vec{L}_2 \Rightarrow \vec{W}_2 \times \vec{L}_2 = \vec{T}_2 \\ \xi_3 \vec{W}_3 \times \vec{L}_3 &= \sin\theta_3 \vec{W} \times \vec{L}_3 \Rightarrow \vec{W}_3 \times \vec{L}_3 = \vec{T}_3 \end{aligned}\right\} . \tag{2-54}$$

Thus, the lines $\vec{T}_i$ are the common perpendicular to the lines $\vec{W}_i$ and $\vec{L}_i$. The same property exists for $\vec{F}_i$ and $\vec{L}_i$ whose common perpendicular is $\vec{\Gamma}_i$.

Therefore (disregarding the orientation of trihedrons) we have

$$\text{and} \qquad \left.\begin{aligned} \vec{W}_i &\equiv \vec{\Gamma}_i \\ \vec{F}_i &\equiv \vec{T}_i \end{aligned}\right\} . \tag{2-55}$$

This menas $\{\vec{L}_i, \vec{T}_i, \vec{\Gamma}_i\} \equiv \{\vec{L}_i, \vec{W}_i, \vec{F}_i\}$.

Hence, if the configuration of acceleration axes is given, the distributions of $\vec{W}$ and $\vec{F}$ are (disregarding a change of time unit) completely determined.

When $\vec{V}_i$ are given $\vec{L}_i$ can be constructed and $\vec{T}_i \equiv \vec{F}_i$ also according to (2-50) can be erected; $\vec{\Gamma}_i$ is the third orthogonal line of the rectangular trihedron $\{\vec{L}_i, \vec{T}_i, \vec{\Gamma}_i\}$.

The common perpendicular to the three lines $\vec{T}_1, \vec{T}_2, \vec{T}_3$ is $\vec{W}$ and to the three lines $\vec{\Gamma}_1, \vec{\Gamma}_2, \vec{\Gamma}_3$ is $\vec{F}$.

Eventually, at the instant t, let us chose in the space H all lines $\vec{X} = \vec{x} + \varepsilon \vec{x}^*$ which cut the three acceleration axes $\vec{V}_1, \vec{V}_2, \vec{V}_3$. Then we have the following relations for the six Plückerian line coordinates of $\vec{X}$.

Since $\vec{X}$ is a unit dual vector

$$\vec{x}^2 = 1 \quad \text{and} \quad \vec{x} \cdot \vec{x}^* = 0.$$

Since $\vec{X}$ cuts the axes $\vec{V}_1, \vec{V}_2, \vec{V}_3$,

$$\vec{v}_i \cdot \vec{x}^* + \vec{v}^*_i \cdot \vec{x} = 0 \quad (i=1,2,3).$$

Thus we can express the lines $\vec{x}$ by one real parameter t as follows:

$$\vec{X} = \vec{X}(t) = \vec{x}(t) + \varepsilon \vec{x}^*(t)$$

and let us suppose that $\vec{X} = \vec{X}(t)$ is differentiable. Then it is a ruled surface.

III. SPECIAL CASES

A. The case of $\alpha = 0$ *and* $\alpha^* = 0$:

In this case the determinant D vanishes; in other words $\vec{\Psi}'$ and $\dot{\vec{\Psi}}'$ are linearly dependent. Therefore $\vec{W}$ and $\vec{F}$ correspond to the same line l in the space and the points of l have no acceleration. Hence for the locus of points with zero-acceleration the velocity is a constant vector, therefore according to {(3-10), [2]} if the corresponding unit dual vector is $\vec{A}$, we may write

$$\frac{d_f\vec{A}}{dt} = \vec{V}_0 \tag{3-1}$$

where $\vec{V}_0$ is a constant dual vector, and by integration we obtain

$$\vec{A} = \vec{A}_0 + \vec{V}_0\,(t-t_0)$$

$$\vec{A} = \vec{a}_0 + \varepsilon\,\vec{a}_0^* + (t-t_0)\,(\vec{v}_0 + \varepsilon\vec{v}_0^*)$$

$$\vec{A} = \vec{a}_0 + (t-t_0)\,\vec{v}_0 + \varepsilon\,[\vec{a}^*_0 + (t-t_0)\,\vec{v}^*_0] \tag{3-2}$$

where $\vec{A}_0$ is the initial constant vector and where $\vec{A}$ is a unit dual vector:

$$\vec{A}^2 = 1. \tag{3-3}$$

The unit dual vector $\vec{A}$ with a real parameter t represents a differentiable family of straight lines in the three dimensional fixed space H′. This means the locus of points with zeo-acceleration is a ruled surface. The lines $\vec{A}(t)$ are the generators or rulings of the surface and at the instant t of the motion H/H′ this unit dual vector corresponds to the line l.

Now we are going to discuss the properties of the orbit surface of l (ruled surface of l). During the motion H/H′ the unit dual vector $\vec{A}$ draws a curve as its dual spherical representation on the unit dual sphere. If the dual arc length of this curve is dS then

$$dS = ds + \varepsilon ds^* \qquad (3\text{-}4)$$

and

$$dS^2 = (d_f\vec{A})^2$$

$$dS^2 = (\vec{V}_0)^2 = (\vec{v}_0 + \varepsilon \vec{v}_0^{\,*})^2$$

$$dS^2 = \vec{v}_0^{\,2} + 2\,\varepsilon\, \vec{v}_0 \cdot \vec{v}_0^{\,*} \qquad (3\text{-}5)$$

the "drall" of the ruled surface $\vec{A} = \vec{A}\,(t)$ is

$$\frac{1}{d} = \frac{ds\ ds^*}{ds^2} \qquad (3\text{-}6)$$

and according to (3-5) the drall is

$$\frac{1}{d} = \frac{\vec{v}_0 \cdot \vec{v}^*_0}{\vec{v}_0^{\,2}}\,. \qquad (3\text{-}7)$$

Therefore we conclude that the ruled surface (l) has a constant drall. Hence we have following theorem [3].

Theorem 3-1. In the motion H/H′, at the instant t the line which has no acceleration is included in a square line complex of H.

As a special case if $\vec{V}_0$ is a unit dual vector then $\vec{v}_0 \cdot \vec{v}^*_0 = 0$ and $\frac{1}{d} = 0$. Therefore, the ruled surface (l) itself is developable. In this case the dual spherical representation curve has the real arc length dS=ds. On the other hand $\vec{A}$ (t) and its neighboring $\vec{A}$ (t+dt) meet on the edge of regression of the ruled surface (l), i. e. the tangent lines of the edge of regression are the lines $\vec{A}$. Thus we have the following theorem:

Theorem 3-2. If the lines $\vec{A}$ of H generate a developable surface in H′ then $\vec{A}$ is included in a special square line complex which is identical to the complex of the tangent lines of orbits of ∞^3 points of H.

B. The case of $\alpha = 0,\ \alpha^* \neq 0$:

In this case also the determinant D vanishes, so that $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are linearly dependent, but the lines $\vec{W}$ and $\vec{F}$ are just parallel, they are not coincident; their minimal distance is $\alpha^* \neq 0$.

Since the corresponding unit dual vectors are $\vec{W}$ and $\vec{F}$, the accelerations of the points on $\vec{W}$ and $\vec{F}$ are zero so the velocities of these points are constants. Hence denoting the dual constant velocity vectors by $\vec{V}_0$ and $\vec{Y}_0$ we may write

$$\left.\begin{aligned} \frac{d_f\vec{W}}{dt} &= \vec{V}_0 \\ \frac{d_f\vec{F}}{dt} &= \vec{Y}_0 \end{aligned}\right\} \qquad (3\text{-}8)$$

and therefore by integration we obtain

$$\left.\begin{aligned} \vec{W} &= \vec{W}_0 + \vec{V}_0 \quad (t\text{-}t_0) \\ \vec{F} &= \vec{F}_0 + \vec{Y}_0 \quad (t\text{-}t_0) \ . \end{aligned}\right\} \qquad (3\text{-}9)$$

If at the instant t a line $\vec{X}$ of the moving space H intersects the lines $\vec{W}$ and $\vec{F}$ then we have

$$\left.\begin{aligned} \vec{w} \cdot \vec{x}^* + \vec{w}^* \cdot \vec{x} = 0 \\ \vec{f} \cdot \vec{x}^* + \vec{f}^* \cdot \vec{x} = 0 \end{aligned}\right\} \quad (3\text{-}10)$$

In the space H we can find ∞^2 such lines $\vec{X}$ and these lines $\vec{X}$ form a *linear line congruence.* $\vec{W}$ and $\vec{F}$ are the *principal directions of* this linear line congruence.

The discriminant of this linear line congruence is

$$D = (\vec{w} . \vec{w}^*) \quad (\vec{f} . \quad \vec{f}^*) \; - \; (\vec{w} . \vec{f}^* + \vec{w}^* . \; \vec{f})^2 . \quad (3\text{-}11)$$

Since $\vec{W}$ and $\vec{F}$ are unit dual vectors the result is

$$D < 0 . \quad (3\text{-}12)$$

Thus this congruence is a *hyperbolic* congruence {[4]; p. 248}.

C. The case of $\alpha = \frac{\pi}{2}, \; \alpha^* = 0$:

In this case from {(3-17), [2]}

$$D = - \Psi'^2 \quad \dot{\Psi}'^2 . \quad (3\text{-}13)$$

The lines $\vec{W}$ and $\vec{F}$ meet at a right angle, the accelerations of the points on these two orthogonal lines are not zero. For the acceleration axes, {(3-21), [2]} reduces

$$\Lambda^3 - \Lambda^2 + K\Lambda = 0 . \quad (3\text{-}14)$$

And therefore

$$\Lambda_1 = 0,$$

hence , the cubic curve f = 0 in the (k, $\cos^2\alpha$) – plane becomes

$$f \equiv k\ (4k\text{-}1) = 0 \tag{3-15}$$

and it shows that there are two parallel lines $k = 0$ and $k = \frac{1}{4}$.
The cusp point and the asymptotes of (C) have disappeared. The condition of reality of the acceleration axes is, from {(3-29), [2]},

$$f \equiv k\ (4k-1) \leqq 0\ . \tag{3-16}$$

In the configuration, $\Lambda_1 = 0$ and (2-6) give

$$\cos^2\Theta_1 = 0 \tag{3-17}$$

and therefore from (2-14) we obtain

$$\cos\varnothing_{12} = \cos\varnothing_{13} = 0\ . \tag{3-18}$$

This means that the acceleration axis $\vec{V}_1$ isthe common perpendicular to three lines $\vec{V}_2$, $\vec{V}_3$ and $\vec{W}$. On the other hand the equality (2-32) gives

$$\cos\Sigma_2 = \cos\Sigma_3 = 0\ . \tag{3-19}$$

This means that the common perpendicular to $\vec{V}_2$ and $\vec{V}_3$ is $\vec{F}$. Thus the lines $\vec{V}_1$ and $\vec{F}$ are coincident, i. e:

$$\vec{V}_1 \equiv \vec{F}\ .$$

From (3-19) and (2-38) the result is

$$\sin Q_2 = \sin Q_3 = 0 \tag{3-20}$$

and since $\vec{F} = \vec{V}_1$ i. e. $\Sigma_1 = 0$,

$$\sin Q_1 = 1\ . \tag{3-21}$$

Hence we conclude that the lines $\vec{F}_2$, $\vec{F}_3$ and $\vec{F}$ are coincident and the lines $\vec{F}_1$ and $\vec{F}$ meet at a right angle. Thus the lines $\vec{F} \equiv \vec{V}_1$ is the common perpendicular to the four lines $\vec{V}_2$, $\vec{V}_3$, $\vec{W}$ and $\vec{F}_1$.

From (2-41) and (3-17) we obtain that

$$\sin P_1 = 0 \ .$$

This means that the lines $\vec{W}$ and $\vec{W}_1$ are coincident.

The illustration of configuration Fig. (2-4) reduces to Fig. (3-1).

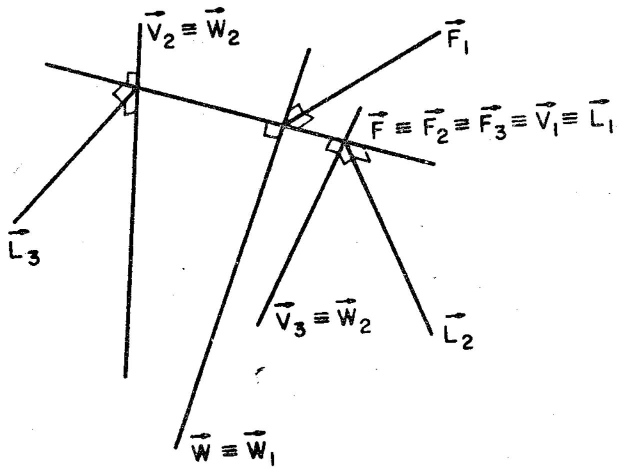

FIG. 3-1

$\vec{L}_1 \equiv \vec{F}$;

$\vec{L}_2$ is the normal of the $(\vec{V}_3, \vec{F})$ – plane;

$\vec{L}_3$ is the normal of the $(\vec{V}_2, \vec{F})$ – plane;

$\vec{F}_1$ is the normal of the $(\vec{L}_1, \vec{W})$ – plane.

Hence, in this special case, when the configuration of the acceleration axes is given, the common perpendiculra $\vec{V}_2$ and $\vec{V}_3$ is $\vec{F}$ and $\vec{W} \equiv \vec{W}_1$.

Hence we may express the following theorem in conclusion:

Theorem 3-3. In the motion H/H′ if the lines $\vec{W}$ and $\vec{F}$ meet at a right angle, then $\vec{F}$ is coincindent with the lines V_i, L_i, $\vec{F}_j$, $\vec{F}_k$ and it is the common perpendicular to the six lines $\vec{F}_i$, $\vec{V}_j$, $\vec{V}_k$, $\vec{L}_j$, $\vec{L}_k$, $\vec{W}$.

D. The case of $\alpha = \dfrac{\pi}{2}$, $\alpha^* \neq 0$:

In this case also D verifies the equation (3-13). From {(3-21), [2]} we have

$$\Lambda_1 = 0 \quad \text{or} \quad \cos\Theta_1 = 0,$$

and according to (2-37) and (2-41) it follows that $\sin Q_3 = 0$ and $\sin P_1 = 0$ respectively. Since $\cos\nabla = -\varepsilon\alpha^* \neq 0$ these two reusults do not satisfy the equation (2-43) so this special case does not exist.

Thus we see that the lines $\vec{W}$ and $\vec{F}$ specially can be coincident, parallel and can intersect orthogonally but they cannot be skew orthogonal.

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ÖZET

Bu çalışmada genel olarak, ivme eksenlerinin geometrik özelikleri ve bu eksenlere karşılık gelen hız ve ivme dağılımları ele alındı. Ayrıca reel küresel halin dışında kalan özel haller de eleştirildi.

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