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by

Erdoğan ÇITAKOĞLU

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Faculté des Sciences de l'Université d'Ankara
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ABSTRACT

Heat transfer from a nuclear reactor solid fuel tube with a constant internal energy generation to a coolant in slug flow is studied analytically and the temperature fields in the fuel tube and coolant are determined in the forms of a series solutions. The physical properties, heat transfer coefficient and thermal contact conductance are assumed to be constant.

NOMENCLATURE

A	:	Cross-sectional area of the coolant channel
A_n	:	Dimensionless numbers
α_n	:	Dimensionless coefficients
B_n	:	Dimensionless coefficients
β_n	:	Angle
C_n	:	Dimensionless coefficients
c_p	:	Specific heat of the coolant at constant pressure
D_n	:	Dimensionless coefficients
δ	:	Thickness of the cladding
δ_n	:	Dimensionless coefficients
Δ_n	:	Dimensionless coefficients
E	:	Dimensionless coefficient
F_n	:	Dimensionless coefficients
H_n	:	Dimensionless coefficients
$\eta = r/R_1$	:	Dimensionless coordinate in radial direction
h	:	Heat transfer coefficient between the cladding and the coolant
I_0, I_1	:	Zeroth and First order modified Bessel functions of the first kind
K_0, K_1	:	Zeroth and First order modified Bessel functions of the second kind
k	:	Thermal conductivity
L	:	Lenght of the rod, Boundary
λ_n	:	Characteristic values
m	:	Dimensionless constant

n	:	Dimensionless constant
P	:	Periphery of the coolant channel
φ	:	Dimensionless temperature
φ_n	:	Element of the non-orthogonal set of functions
Ψ	:	Dimensionless temperature
Ψ_n	:	Element of the orthonormal set of functions
q'''	:	Internal energy generation per unit time and volume

$q'''^* = \dfrac{q''R_1^2}{k_s T_0}$: Dimensionless internal energy generation

R	:	Radius
r	:	Coordinate in radial direction
ρ	:	Density of the coolant
T	:	Temperature

$T^* = \dfrac{T - T_0}{T_0}$: Dimensionless temperature

u	:	Thermal conductance between the fuel rod and the cladding
v	:	Velocity of the coolant
Y	:	Dimensionless temperature
Z	:	Dimensionless temperature
z	:	Coordinate in axial direction
$\zeta = z/R_1$	:	Dimensionless coordinate in axial direction

SUBSCRIPTS

0 : Inner radius of the cladding, Inlet condition ($z = 0$)
1 : Inner radius of the fuel tube.
2 : Outer radius of the fuel tube
s : Solid
c : Cladding
f : Fluid

INTRODUCTION

Heat transfer between the pipe or rod and fluid flowing in axial direction with special boundary conditions has a wide application in practice. When internal energy is generated in pipe or rod the problem finds its application in nuclear engineering. Many research works have been published dealing with various aspects of the problem. For instance Fagan and Leipziger [1] worked with heat generating solid circular cylinder or sphere cooled in steady state with arbitarily varying heat transfer coefficient on the surface and simple approximate solutions have been obtained for the two cases where the heat transfer coefficient differs greatly from its average value and it does not differ much from its average value. The author [2, 3] solved the heat transfer problem from a solid circular rod with internal energy generation into the fluid flowing parallel to the axis of the rod in steady state. For the different geometries of the coolant channel we can mention the work of Schmidt and Newell [4] which dealt with heat transfer in fully developed laminar flow through rectangular and isosceles triangle ducts with various aspect ratios under two specific boundary conditions; constant wall temperature and constant heat flux. Pearson [5] found the temperature distribution for so-called "inscribable" non circular ducts having constant wall heat flux and slug flow. In the design of the fuel tubes in nuclear engineering the axial conduction in the tube usually assumed to be negligible [6, 7]. The author solved the heat transfer problem from a fuel tube with uniform internal energy generation to a coolant flowing coaxially in slug flow taking into account the conduction in the axial direction in the tube.

FORMULATION OF THE PROBLEM

The core is made up of a single solid moderator block and fuel tubes pass through the moderator (Fig. 1). The coolant which flows upward in fuel tubes removes the internal energy generated in the fuel tubes.

One typical fuel tube with the coolant is shown in Fig. 2. Inner and outer radius of the tube are indicated with R_1 and R_2, and the lenght with L. The inner surface of it is covered with cladding of thickness δ and the thermal contact conductance u between the tube and cladding is assumed not to be negligible. The heat transfer coefficient between the cladding and the coolant is h. The radius, cross sectional area and periphery of the coolant channel are denoted by R_0, A and P respectively.

The following assumptions are made:

a) The fuel tube has a uniform rate of internal energy generation per unit volume, and is insulated at outer surface and at both ends.

b) The heat transfer through the cladding, which is considered as a thin slab, is in radial direction only.

c) The coolant has a slug flow with uniform velocity v. The temperature in the coolant is uniform in radial direction and the axial conduction is neglected.

d) The physical properties, heat transfer coefficient and thermal contact conductance are constant.

The reference frame is chosen as shown in Fig. 2. The origin is at the center of the lower end of the fuel tube and the coordinate axis r and z are in the radial and vertical directions respectively. We then have the following.

Governing equation for the fuel tube :

$$k_s\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right)+\frac{\partial^2 T_s}{\partial z^2}\right]+q'''=0 \qquad \text{(Eq.1)}$$

where k_s and $T_s(r, z)$ are the thermal conductivity and temperature field of the fuel tube, q''' shows the uniform rate of internal energy generation of the fuel tube per unit volume.

Boundary conditions :

$$k_s\frac{\partial T_s(R_1, z)}{\partial r}=u\,[T_s(R_1, z)-T_c(R_1, z)], \qquad \text{(Eq.2)}$$

$$\frac{\partial T_s(R_2, z)}{\partial r}=0, \qquad \text{(Eq.3)}$$

$$\frac{\partial T_s(r, 0)}{\partial z}=0 \qquad \text{(Eq.4)}$$

$$\frac{\partial T_s(r, L)}{\partial z}=0 \qquad \text{(Eq.5)}$$

where $T_c(r, z)$ is the cladding temperature.

Heat transfer through the cladding:

$$u\,[T_s(R_1, z) - T_c(R_1, z)] = k_c \frac{T_c(R_1, z) - T_c(R_0, z)}{\delta}$$

$$= h\,[T_c(R_0, z) - T_f(z)] \qquad \text{(Eq.6)}$$

where $T_f(z)$ is the coolant temperature.

Governing equation for the coolant:

$$\rho c_p\,Av \frac{dT_f}{dz} = Ph\,[T_c(R_0, z) - T_f(z)] \qquad \text{(Eq.7)}$$

where ρ is the density and c_p is the specific heat of the coolant at constant pressure.

Boundary condition:

$$T_f(0) = T_0 \qquad \text{(Eq.8)}$$

where T_0 is the inlet temperature of the coolant.

$T_c(R_0, z)$ and $T_c(R_1, z)$ can be expressed in terms of $T_f(z)$ and $T_s(R_0, z)$ by the use of (Eq.6) and are obtained:

$$T_c(R_1, z) = (1-m)\,T_s(R_1, z) + m\,T_f(z) \qquad \text{(Eq.9)}$$

$$T_c(R_0, z) = n\,T_s(R_1, z) + (1-n)\,T_f(z) \qquad \text{(Eq.10)}$$

where,

$$m \equiv \frac{hk_c/\delta}{(h+k_c/\delta)\,u+hk_c/\delta} \qquad \text{(Eq.11)}$$

$$n \equiv \frac{uk_c/\delta}{(h+k_c/\delta)\,u+hk_c/\delta} \qquad \text{(Eq.12)}$$

The coefficients m and n are dimensionless numbers.

Substitution of (Eq. 9) into (Eq. 2) yields,

$$k_s \frac{\partial T_s(R_1, z)}{\partial r} + u\, m\, T_s(R_1, z) = u\, m\, T_f(z) \qquad \text{(Eq.13)}$$

This change reduces the problem to solving one partial differential equation (Eq.1) and one ordinary differential equation (Eq.7) with the boundary conditions (Eq.13, 3, 4, 5) and (Eq.8). The equation (Eq.13) stands for the boundary condition (Eq.2) and the heat transfer equation through the cladding (Eq.6).

The problem can be formulated in the following dimensionless form:

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial T_s^*}{\partial \eta}\right) + \frac{\partial^2 T_s^*}{\partial \zeta^2} + q'''^* = 0 \qquad \text{(Eq.14)}$$

with,

$$\frac{\partial T_s^*(1, \zeta)}{\partial \eta} + \frac{R_1 um}{k_s} T_s^*(1, \zeta) = \frac{R_1 um}{k_s} T_f^*(\zeta) \qquad \text{(Eq.15)}$$

$$\frac{\partial T_s(R_2/R_1, \zeta)}{\partial \eta} = 0, \qquad \text{(Eq.16)}$$

$$\frac{\partial T_s^*(\eta, 0)}{\partial \zeta} = 0 \qquad \text{(Eq.17)}$$

$$\frac{\partial T_s^*(\eta, L/R_1)}{\partial \zeta} = 0, \qquad \text{(Eq.18)}$$

and from (Eq.2)

$$\frac{dT_f^*(\zeta)}{\partial \zeta} = \frac{Pk_s}{\rho c_p Av} \frac{dT_s^*(1, \zeta)}{\partial \eta} \qquad \text{(Eq.19)}$$

with

$$T_f^*(0) = 0. \qquad \text{(Eq.20)}$$

The associated dimensionless variables are defined as follows:

$$\eta \equiv \frac{r}{R_1}, \qquad T_s^* \equiv \frac{T_s - T_0}{T_0}, \qquad \zeta \equiv \frac{z}{R_1},$$

$$q'''^* \equiv \frac{q''' R_1^2}{k_s T_0} \quad \text{and} \quad T_f^* \equiv \frac{T_f - T_0}{T_0}$$

SOLUTION OF THE PROBLEM

We now separate non-homogeneous partial differential equation (Eq. 14) into one homogeneous partial differential equation which will form a characteristic value problem with two homogeneous and one homogeneous and one non-homogeneous boundary conditions obtainable from (Eq. 15, 16, 17) and (18), and one second order ordinary differential equation. We assume a temperature field in the following form,

$$T_s^*(\eta, \zeta) = \psi(\eta, \zeta) + \varphi(\eta). \qquad \text{(Eq.21)}$$

The characteristic value problem becomes now

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \psi}{\partial \eta}\right) + \frac{\partial^2 \psi}{\partial \zeta^2} = 0, \qquad \text{(Eq.22)}$$

with the boundary conditions

$$\frac{\partial \psi(1, \zeta)}{\partial \eta} + \frac{R_1 um}{k_s} \psi(1, \zeta) = \frac{R_1 um}{k_s} T_f^*(\zeta) - \frac{d\varphi(1)}{d\eta} - \frac{R_1 um}{k_s} \varphi(1) \qquad \text{(Eq.23)}$$

$$\frac{\partial \psi(R_2/R_1, \zeta)}{\partial \eta} = 0 \qquad \text{(Eq.24)}$$

$$\frac{\partial \psi(\eta, 0)}{\partial \zeta} = 0 \qquad \text{(Eq.25)}$$

$$\frac{\partial\psi(\eta, L/R_1)}{\partial\zeta} = 0. \qquad \text{(Eq.26)}$$

The other ordinary differential equation assumes,

$$\frac{1}{\eta}\frac{d}{d\eta}(\eta\frac{d\varphi}{d\eta}) + q'''^{*} = 0, \qquad \text{(Eq.27)}$$

with the boundary condition

$$\frac{d\eta(R_2/R_1)}{d\eta} = 0. \qquad \text{(Eq.28)}$$

We can readily integrate (Eq.27) twice and obtain

$$\varphi(\eta) = -\frac{q'''^{*}}{4}\eta^2 + A_1 \ln \eta + A_2 \qquad \text{(Eq.29)}$$

The boundary condition (Eq.28) yields $A_1 = \frac{q'''^{*}}{2}(\frac{R_2}{R_1})^2$ and the other integration constant is taken arbitrarily zero, thus $A_2 = 0$. Therefore (Eq.29) reads

$$\varphi(\eta) = -\frac{q'''^{*}}{4}\eta^2 + \frac{q'''^{*}}{2}(\frac{R_2}{R_1})^2 \ln \eta. \qquad \text{(Eq.30)}$$

We now apply the separation of variables for the solution of (Eq.22) and assume a solution in the following form,

$$\psi(\eta, \zeta) = Y(\eta)\, Z(\zeta). \qquad \text{(Eq.31)}$$

The differential equation, when separated, becomes,

$$\frac{1}{\eta Y}\frac{d}{d\eta}(\eta\frac{dY}{d\eta}) = -\frac{1}{Z}\cdot\frac{d^2Z}{d\zeta^2} = \lambda^2$$

where λ is the separation constant and the sign of λ^2 is chosen positive in order to make the homogeneous direction ζ a characteristic direction.

The differential equation associated with the non-homogeneous direction η is,

$$\frac{d}{d\eta}\left(\eta\frac{dY}{d\eta}\right) - \lambda^2 \eta Y = 0, \qquad \text{(Eq.33)}$$

and the boundary condition (Eq.24) reduces to

$$\frac{dY(R_2/R_1)}{d\eta} = 0. \qquad \text{(Eq.34)}$$

We leave the non-homogeneous boundary condition (Eq.23) to the end. The general solution of (Eq.33) then is

$$Y(\eta) = B_1 I_0(\lambda\eta) + B_2 K_0(\lambda\eta), \qquad \text{(Eq.35)}$$

where I_0 and K_0 are the zeroth order modified Bessel functions of the first and second kind respectively and B_1 and B_2 are arbitrary constants.

From the boundary condition (Eq.34) we find,

$$B_2 = \frac{I_1(\lambda R_2/R_1)}{K_1(\lambda R_2/R_1)} B_1 ,$$

where I_1 and K_1 are first order modified Bessel functions of the first and second kind respectively. The general solution (Eq.35) then becomes:

$$Y(\eta) = B_1 \left[I_0(\lambda\eta) + \frac{I_1(\lambda R_2/R_1)}{K_1(\lambda R_2/R_1)} K_0(\lambda\eta)\right]. \qquad \text{(Eq.36)}$$

The differential equation associated with the characteristic direction is

$$\frac{d^2Z}{d\zeta^2} + \lambda^2 Z = 0, \qquad \text{(Eq.37)}$$

with the following boundary conditions obtained from (Eq.25–26),

$$\frac{dZ(0)}{d\zeta} = 0 \qquad \text{(Eq.38)}$$

and

$$\frac{dZ(L/R_1)}{d\zeta} = 0. \qquad \text{(Eq.39)}$$

The general solution of (Eq.37) is

$$Z(\varsigma) = C_1 \cos \lambda\xi + C_2 \sin \lambda\xi, \qquad \text{(Eq.40)}$$

where C_1 and C_2 are arbitrary constants to be determined from the boundary conditions (Eq.38–39). From the (Eq.38) we obtain $C_2 = 0$ and the other one (Eq.39) yields $\sin(\lambda L/R_1) = 0$ which is satisfied for the set of values of λ,

$$\lambda_n = \frac{\pi n}{L/R_1} . \qquad \text{(Eq.41)}$$

The associated characteristic functions are

$$Z_n(\zeta) = \cos \frac{\pi n}{L/R_1} \zeta . \qquad \text{(Eq.42)}$$

Therefore, the general solution of the characteristic value problem, in the form of infinite series, is,

$$\psi(\eta, \varsigma) = D_0 + \sum_{n=1}^{\infty} D_n \left[I_0\left(\frac{\pi n}{L/R_1} \eta\right) + \frac{I_1\left(\frac{\pi n}{L/R_2}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_0\left(\frac{\pi n}{L/R_1} \eta\right)\right] \cos \left(\frac{\pi n}{L/R_1} \zeta\right), \qquad \text{(Eq.43)}$$

where D_n are arbitrary constants to be determined from the differential equation (Eq.19) and the boundary conditions (Eq.20) and (23). While taking the first constant D_0 out of the summation symbol in which n = 0, the property $\lim_{\alpha=0} \frac{I_1(\alpha)}{K_1(\alpha)} K_0(\alpha) = 0$ is used.

The temperature field in the fuel tube is obtained from (Eq.21, 30) and (43),

$$T_s^*(\eta, \zeta) = D_0 + \sum_{n=1}^{\infty} D_n [I_0 (\frac{\pi n}{L/R_1} \eta) +$$

$$+ \frac{I_1(\frac{\pi n}{L/R_2})}{K_1(\frac{\pi n}{L/R_2})} K_0 (\frac{\pi n}{L/R_1} \eta)] \cos(\frac{\pi n}{L/R_1} \zeta) - \frac{q'''^*}{4} \eta^2 + \frac{q'''^*}{2} (\frac{R_2}{R_1})^2 \ln \eta. \quad \text{(Eq.44)}$$

To determine the coefficients D_n in (Eq. 43) and (44), we first differentiate the boundary condition (Eq.23) with respect to ζ,

$$\frac{\partial^2 \psi(1, \zeta)}{\partial \zeta \, \partial \eta} + \frac{R_1 um}{k_s} \frac{\partial \psi(1, \zeta)}{\partial \zeta} = \frac{R_1 um}{k_s} \frac{dT_f^*(\zeta)}{d\zeta}. \quad \text{(Eq.45)}$$

Differentiating $T_s^*(\eta, \zeta)$ with respect to η in (Eq.21) and using (Eq.30) we obtain for $\eta = 1$,

$$\frac{\partial T_s^*(1, \zeta)}{\partial \eta} = \frac{\partial \psi(1, \zeta)}{\partial \eta} + \frac{q'''^*}{2} [(\frac{R_2}{R_1})^2 - 1]. \quad \text{(Eq.46)}$$

From (Eq.19) and (46) we can write

$$\frac{dT_f^*(\zeta)}{d\zeta} = \frac{Pk_s}{\rho c_p Av} \frac{\partial \psi(1, \zeta)}{\partial \eta} + \frac{Pk_s q'''^*}{2\rho \, c_p Av} [(\frac{R_2}{R_1})^2 - 1]. \quad \text{(Eq.47)}$$

Eliminating $dT_f(\zeta)/d\zeta$ between (Eq.45) and (47) and rearranging we obtain,

$$\frac{\partial^2\psi(1,\zeta)}{\partial\zeta\,\partial\eta} + \frac{R_1 um}{k_s}\frac{\partial\psi(1,\zeta)}{\partial\zeta} - \frac{R_1 umP}{\rho c_p\,Av}\frac{\partial\psi(1,\zeta)}{\partial\eta} =$$

$$= \frac{R_1 um\,Pq'''^*}{2\rho C_p\,Av}\left[\left(\frac{R_2}{R_1}\right)^2 - 1\right]. \qquad \text{(Eq.48)}$$

The partial derivatives of $\psi(\eta, \zeta)$ are obtained from (Eq.43)

$$\frac{\partial\psi(\eta,\zeta)}{\partial\eta} = \sum_{n=1}^{\infty} D_n\left(\frac{\pi n}{L/R_1}\right)\left[I_1\left(\frac{\pi n}{L/R_1}\eta\right) - \right.$$

$$\left. - \frac{I_1\left(\frac{\pi n}{L/R_1}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_1\left(\frac{\pi n}{L/R_1}\eta\right)\right] \cos\left(\frac{\pi n}{L/R_1}\zeta\right),$$

$$\frac{\partial\psi(\eta,\zeta)}{\partial\zeta} = -\sum_{n=1}^{\infty} D_n\left(\frac{\pi n}{L/R_1}\right)\left[I_0\left(\frac{\pi n}{L/R_1}\eta\right) + \right.$$

$$\left. + \frac{I_1\left(\frac{\pi n}{L/R_2}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_1\left(\frac{\pi n}{L/R_1}\eta\right)\right] \sin\left(\frac{\pi n}{L/R_1}\zeta\right),$$

$$\frac{\partial^2\psi(\eta,\zeta)}{\partial\zeta\,\partial\eta} = -\sum_{n=1}^{\infty} D_n\left(\frac{\pi n}{L/R_1}\right)^2\left[I_1\left(\frac{\pi n}{L/R_1}\eta\right) - \right.$$

$$\left. - \frac{I_1\left(\frac{\pi n}{L/R_2}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_1\left(\frac{\pi n}{L/R_1}\eta\right)\right] \sin\left(\frac{\pi n}{L/R_1}\zeta\right).$$

Subtituting these partial derivatives for $\eta = 1$ into (Eq.48) we obtain,

$$\sum_{n=1}^{\infty} D_n [a_n \cos(\frac{\pi n}{L/R_1} \zeta) + b_n \sin(\frac{\pi n}{L/R_1} \zeta)] = - E \qquad \text{(Eq.49)}$$

where,

$$a_n = \frac{R_1 \, um \, P}{\rho \, c_p \, Av} (\frac{\pi n}{L/R_1}) [I_1 (\frac{\pi n}{L/R_1}) - \frac{I_1 (\frac{\pi n}{L/R_2})}{K_1(\frac{\pi n}{L/R_2})} K_1 (\frac{\pi n}{L/R_1})], \qquad \text{(Eq.50)}$$

$$b_n = \frac{\pi n}{L/R_1} \{ \frac{\pi n}{L/R_1} [I_1 (\frac{\pi n}{L/R_1}) - \frac{I_1 (\frac{\pi n}{L/R_2})}{K_1(\frac{\pi n}{L/R_2})} K_1 (\frac{\pi n}{L/R_1})] +$$

$$+ \frac{R_1 um}{k_s} [I_0 (\frac{\pi n}{L/R_1}) + \frac{I_1 (\frac{\pi n}{L/R_2})}{K_1(\frac{\pi n}{L/R_2})} K_0 (\frac{\pi n}{L/R_1})] \}, \qquad \text{(Eq.51)}$$

$$E = \frac{R_1 \, um \, P \, q'''^*}{2 \, \rho \, c_p \, Av} [(\frac{R_2}{R_1})^2 - 1] . \qquad \text{(Eq.52)}$$

We now define an angle β_n as,

$$\tan \beta_n = \frac{a_n}{b_n} . \qquad \text{(Eq.53)}$$

Carrying (Eq.53) into (Eq.49) and rearranging we obtain

$$\sum_{n=1}^{\infty} D_n \frac{b_n}{E \cos \beta_n} \sin \left(\frac{\pi n}{L/R_1} \zeta + \beta_n\right) = -1 . \qquad \text{(Eq.54)}$$

In (Eq.54) b_n and β_n are known parameters as defined by the relations (Eq.50.51) and (53). To determine D_n we have to find the expansion of the constant (—1) in terms of the functions

$\varphi_k = \sin \left(\frac{\pi k}{L/R_1} + \beta_k\right)$ which forms a non-orthogonal infinite set of functions. However, we can form an orthonormal set of functions $\{\psi_n\}$ from the non-orthogonal set of functions $\{\varphi_n\}$. The nth element of the set is,

$$\psi_n(\zeta) = \sum_{k=1}^{n} \alpha_n^{(k)} \sin \left(\frac{k\pi}{L/R_1} \zeta + \beta_k\right), \qquad \text{(Eq.55)}$$

where the values of the coefficients $\alpha_n^{(k)}$ are given in App. (Eq. A5).

We can now represent (—1) in terms of ψ_n in infinite series as,

$$\sum_{n=1}^{\infty} F_n \psi_n(\zeta) = -1 , \qquad \text{(Eq.56)}$$

where the coefficient F_n are given as follows.

$$F_n = -\sum_{k=0}^{e} 2\alpha_n^{(2k+1)} \frac{L/R_1}{(2k+1)\pi} \cos \beta_{2k+1} \qquad \text{(Eq.57)}$$

$$e = \begin{cases} \dfrac{n-1}{2} & \text{when n is odd.} \\[2ex] \dfrac{n-2}{2} & \text{when n is even.} \end{cases}$$

Carrying the values of $\psi_n(\zeta)$ from (Eq. 55) into (Eq. 56) and rearranging we obtain,

$$[\sum_{i=1}^{\infty} F_i \alpha_i^{(1)}] \sin(\frac{\pi}{L/R_1}\zeta+\beta_1) + [\sum_{i=2}^{\infty} F_i \alpha_i^{(2)}] \sin(\frac{2\pi}{L/R_1}\zeta+\beta_2) +$$

$$+\dots+ [\sum_{i=n}^{\infty} F_i \alpha_i^{(n)}] \sin(\frac{\pi n}{L/R_1}\zeta+\beta_n) + \dots = -1. \qquad \text{(Eq.58)}$$

When the coefficients of $\sin(\frac{\pi n}{L/R_1} + \beta_n)$ are compared between the (Eq.58) and (54) we obtain the values of D_n for $n \geq 1$ as

$$D_n = \frac{E \cos \beta_n}{b_n} \sum_{i=n}^{\infty} F_i \alpha_i^{(n)} . \qquad \text{(Eq.59)}$$

We use the boundary condition (Eq. 15) and (20) and the definition (Eq. 51) for the determination of the coefficient D_0 and obtain,

$$D_0 = \frac{R_1 um - [(\frac{R_2}{R_1})^2 - 1]\, 2k_s}{4 R_1 um} q'''^* - \frac{k_s}{R_1 um} \sum_{n=1}^{\infty} D_n b_n (\frac{L/R_1}{\pi n}) .$$

(Eq.60)

Finally substituting the (Eq. 51,57) and (59) into (Eq. 44) we obtain the temperature field $T_s(\eta, \zeta)$ in the fuel tube,

$$T_s^*(\eta, \zeta) = \frac{R_1 um - [(\frac{R_2}{R_1})^2 - 1]\, 2k_s}{4 R_1 um} q'''^* - \frac{q'''^*}{4} \eta^2 +$$

$$+ \frac{q'''^*}{2} (\frac{R_2}{R_1})^2 \ln \eta +$$

$$+ \frac{L\, um\, P\, q'''^*}{\pi \rho\, c_p\, Av} [(\frac{R_2}{R_1})^2 - 1] \sum_{n=1}^{\infty} (\frac{L/R_1}{\pi n}) \cos \beta_n \times$$

$$\times \left\{ \frac{k_s}{R_1 um} - \frac{\left[I_0\left(\frac{\pi n}{L/R_1}\eta\right) + \frac{I_1\left(\frac{\pi n}{L/R_2}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_0\left(\frac{\pi n}{L/R_1}\eta\right)\right] \cos \frac{\pi n}{L/R_1}\zeta}{\frac{\pi n}{L/R_1}\left[I_1\left(\frac{\pi n}{L/R_1}\right) - \frac{I_1\left(\frac{\pi n}{L/R_2}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_1\left(\frac{\pi n}{L/R_1}\right)\right] + \frac{R_1 um}{k_s}\left[I_0\left(\frac{\pi n}{L/R_1}\right) + \frac{I_1\left(\frac{\pi n}{L/R_2}\right)}{K_1\left(\frac{\pi n}{L/R_2}\right)} K_0\left(\frac{\pi n}{L/R_1}\right)\right]} \right\} \times$$

$$\times \sum_{i=n}^{\infty} \alpha_i^{(n)} \sum_{k=0}^{e} \alpha_i^{(2k+1)} \frac{\cos \beta_{2k+1}}{2k+1} . \qquad \text{(Eq. 61)}$$

$$e = \begin{cases} \dfrac{i-1}{2} & \text{when } i \text{ is odd} \\[2ex] \dfrac{i-2}{2} & \text{when } i \text{ is even.} \end{cases}$$

We obtain the temperature field $T_f(\zeta)$ in the coolant by substituting the (Eq. 61) into (15),

$$T_f^*(\zeta) = \frac{k_s P L q'''^*}{\pi \rho c_p A v R_1} \left[\left(\frac{R_2}{R_1}\right)^2 - 1 \right] \sum_{n=1}^{\infty} \left(\frac{L/R_1}{\pi n}\right) \cos\beta_n \left(1 - \cos\frac{\pi n}{L/R_1}\zeta\right) \times$$

$$\times \sum_{i=n}^{\infty} \alpha_i^{(n)} \sum_{k=0}^{e} \alpha_i^{(2k+1)} \frac{\cos \beta_{2k+1}}{2k+1} \qquad \text{(Eq.62)}$$

where

$$e = \begin{cases} \dfrac{i-1}{2} & \text{when } i \text{ is odd} \\[2ex] \dfrac{i-2}{2} & \text{when } i \text{ is even.} \end{cases}$$

DISCUSSION

It is seen that the temperature functions obtained (Eq.60,61) satisfy the differential equations (Eq. 14) and (19) and the appropriate boundary conditions (Eq.15,16,17,18) and (19). The solution can also be checked by assigning special values to the parameters involved. If we assume that the heat generation q''' is zero we have to have the uniform temperature T_0 in the solid and in the fluid because of the physics of the problem. Indeed when we make $q''' = 0$ in the equation (Eq. 61,62) we obtain the constant temperature T_0 for the solid and fluid. If we assume that the coolant velocity v is infinite (or the heat capacity ρc_p is infinite) then the coolant temperature from (Eq. 62) is obtained to be T_0 throughout the coolant, and the fuel temperature from (Eq.61),

$$T_s^*(\eta) = \frac{R_1um - [(\frac{R_2}{R_1})^2 - 1]\, 2k_s}{4\, R_1um}\, q'''^* - \frac{q'''^*}{4}\eta^2 + \frac{q'''^*}{2}(\frac{R_2}{R_1})^2 \ln\eta.$$

Indeed if we solve the boundary-value problem for the same geometry and boundary conditions except the boundary condition on the inside surface of the cladding replaced by the uniform temperature T_0, we obtain the above temperature distribution.

If we now assume $R_2 = R_1$, that corresponds to the reduction of the fuel tube to the circular cylindrical surface through which a constant internal energy generated with the same boundary conditions. In this case we obtain for the coolant temperature T_f to be equal to T_0 throughout the coolant. For the same condition we obtain for the fuel temperature from (Eq. 61),

$$T_s^*(\eta) = \frac{R_1um + 2k_s}{4\, R_1um}\, q'''^* - \frac{q'''^*}{4}\eta^2 + \frac{q'''^*}{2}(\frac{R_2}{R_1}) \ln \eta.$$

This conclusion looks absurd, because in this case the temperature T_s cannot depend on η. The fact is hidden in the solution of this problem. In other words, the circular cylindrical internal energy generating surface and the existing boundary conditions lead to $q''' = 0$. Therefore the condition $R_2 = R_1$ in this problem results in $q''' = 0$, which makes $T_s = T_f = T_0$. A similar reasoning for $L = 0$ leads again to $q''' = 0$.

CONCLUSION

From the temperature distribution function (Eq. 61) we can readily see that the increasing values of dimensional parameters T_0, q''', R_2, and L cause the temperature to increase and again the increasing values of R_1, u, h, k_s, ρc_p, v, and k_c/δ cause the temperature to decrease. When it is compared with the solution of the fuel rod in [2] it is also seen that the increases in T_0, q''' and L cause the temperature to decrease and the increases in ρc_p, v, u, k_s, and k_c/δ cause the temperature to decrease. Further, increase in z and in r cause the fuel tube temperature to increase but, in the case of the fuel rod, increase in r causes the temperature to decrase.

We can also see a similar tendency for the temperature distribution function (Eq .62) in the coolant. Indeed, increasing values of k_s, q''', L and T_0 cause the temperature to increase. On the other hand, when we compare the above results with those of the fuel rod [2], we find out that the temperature increases with increasing values of T_0, q''', and L, and decreases with the increasing values of ρc_p, and v. Further in the increasing direction of the axis z the temperature of the coolant in the fuel tube increases as it is in the case of the fuel rod.

APPENDIX

An orthonormal set of functions $\{\psi_k\}$ which form a complete set can be derived from a given complete non-orthogonal set of functions $\{\varphi_k\}$, [8]. The nth element of the orthonormal set may be written as,

$$\psi_{n+1}(\zeta) = \sum_{k=1}^{n+1} \alpha_{n+1}^{(k)} \varphi_k(\zeta) \qquad \text{(Eq. A1)}$$

or in terms of ψ_k as

$$\psi_{n+1}(\zeta) = \sum_{k=1}^{n} a_{n+1}^{(k)} \psi_k(\zeta) + \alpha_{n+1}^{(n+1)} \varphi_{n+1}(\zeta) . \qquad \text{(Eq. A2)}$$

The orthonormality condition

$$\int_L \psi_{n+1}(\zeta) \psi_k(\zeta) \partial\zeta = \begin{cases} 0 & \text{when } n+1 \neq k \\ 1 & \text{when } n+1 = k \end{cases} \qquad \text{(Eq. A3)}$$

is used to determine the unknown coefficients in (Eq. A2). When these coefficients are carried into (Eq. A2) we obtain the recurrence relation to get the term ψ_{n+1} in terms of ψ_k, for $1 \leq k \leq n$, and φ_{n+1}. This relation may be written in symbolic form as

$$\psi_{n+1}(\zeta) = \frac{\varphi_{n+1}(\zeta) - \sum_{k=1}^{n} \left(\int_L \varphi_{n+1} \psi_k \, d\zeta \right) \psi_k(\zeta)}{\left\{ \int_L \left[\varphi_{n+1}(\zeta) - \sum_{k=1}^{n} \left(\int_L \varphi_{n+1} \psi_k \, d\zeta \right) \psi_k(\zeta) \right]^2 d\zeta \right\}^{1/2}} \qquad \text{(Eq.A4)}$$

When the values of ψ_k, thus obtained, are carried into (Eq. A2) we obtain ψ_{n+1} in terms of φ_k, for $1 \leq k \leq n + 1$, as shown in (Eq. A1). Unfortunately there seems no general term to exist.

The recurrence relation for the coefficients $\alpha_n^{(m)}$ obtained from equation (A4) follows

$$\alpha_1^{(1)} = \frac{1}{(L/2R_0)^{1/2}} . \qquad \text{(Eq. A4)}$$

$\alpha_n^{(m)}$ is defined as the ratio for the sake of simplicity in writing as

$$\alpha_n^{(m)} = \frac{\delta_n^{(m)}}{\Delta n} \qquad \text{(Eq. A6)}$$

for $n \geq 2$ and equation (A.5) for $n = 1$.

The nominator $\delta_n^{(m)}$ and denominator Δ_n are given

$$\delta_n^{(m)} \begin{cases} = - \sum_{k=m}^{n-1} \alpha_k^{(m)} \sum_{i=1}^{k} \alpha_k^{(i)} \int_0^{L/R_0} \varphi_n \varphi_i \, d\zeta, & \text{for } 2 \leq m \leq n-1 \\ = 1 & \text{for } m = n \end{cases} \qquad \text{(Eq.A7)}$$

and,

$$\Delta_n = \left[\frac{L}{2R_0} - \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k} \alpha_k^{(i)} \int_0^{L/R_0} \varphi_n \varphi_i \, d\zeta \right)^2 \right]^{1/2} \qquad \text{(Eq.A8)}$$

for $n \geq 2$, where

$$\int_0^{L/R_0} \varphi_n \varphi_i d\zeta = \frac{L/R_0}{2\pi} \frac{\sin[(n-i)\pi-(\beta_n-\beta_i)]+\sin(\beta_n-\beta_i)}{n-i} -$$

$$- \frac{\sin[(n+i)\pi-(\beta_n+\beta_i)]+\sin(\beta_n+\beta_i)}{n+i}] . \qquad \text{(Eq.A9)}$$

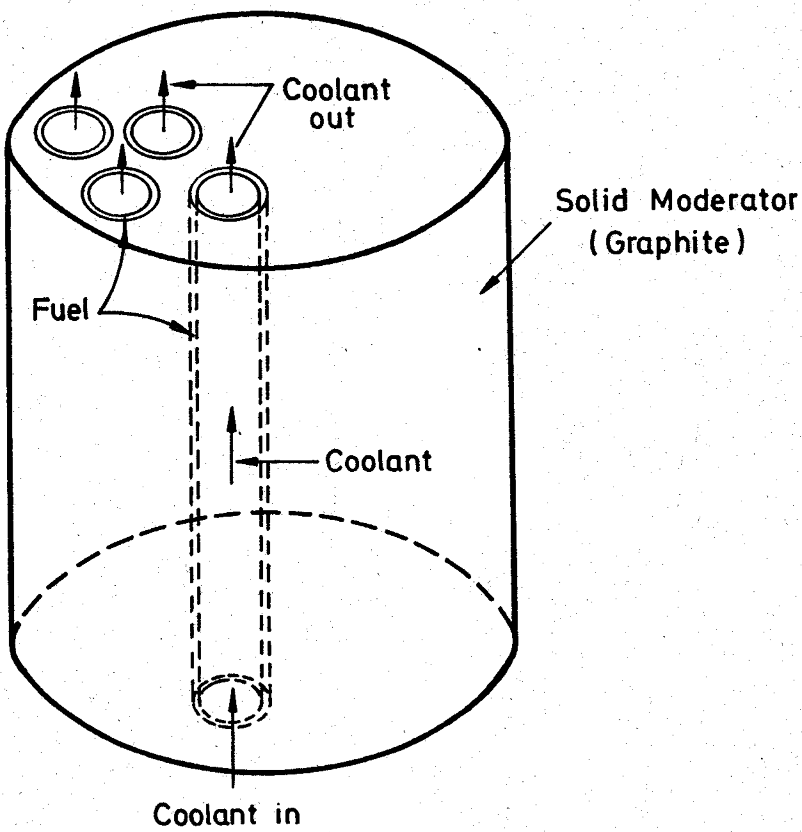

Fig. 1 — Sketch of fuel elements with solid moderator and coolant in a core (Calandria type).

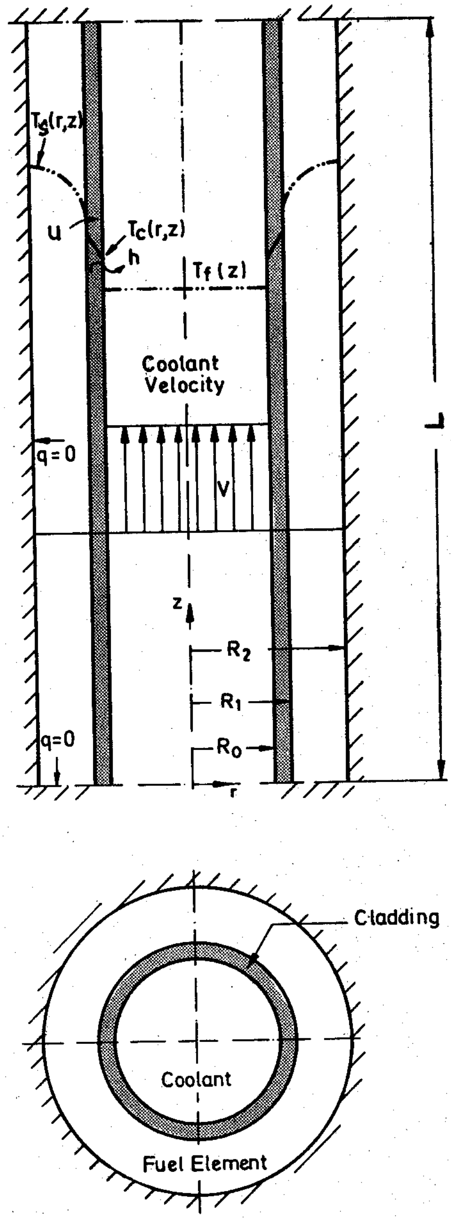

Fig. 2 — **One fuel tube with cladding and coolant.**

REFERENCES

[1] Fagan, W. and Leipziger, S. J. Heat Trans Transc. ASME 88 (1966) 257.
[2] Çıtakoğlu, E., J. Pure and Appl. S., METU. Vol. 3, No. 2, pp. 217–236, 1970.
[3] Çıtakoğlu, E., Promotional Thesis, M.E.T.U., Publication No: 38, 1971.
[4] Schimidt, F. W. and Newell, M. E., Int. J. Heat Mass Trans. 10, 1121 (1967).
[5] Pearson, J. T., Int. J. Heat Mass Trans. 12, 1187 (1969).
[6] Gress King, C. D., Nuclear Power Systems, Mac Millan, 1964.
[7] Glasstone, S. and Sesonske, A., Nuclear Reactor Engineering, Van Nostrand, 1963.
[8] Kantrovich, L. V. and Krylov, V. I., Approximate Methods of Higher Analysis, P. N., 1958.

ÖZET

ÖZEL SINIR ŞARTLARI ALTINDA BİR YAKIT TÜBÜ İÇERİSİNDEKİ SICAKLIK DAĞILIMI

Katı nükleer reaktör yakıt tübü içinden uniform hızla akan soğutucu akışkana ısı geçimi, yakıt içerisinde uniform bir iç enerji üretimi olduğu halde analitik olarak incelenmiş ve yakıt tübü ve soğutucu içindeki sıcaklık alanları serisel olarak bulunmuştur. Fiziksel özellikler, ısı geçimi katsayısı ve ısıl temas katsayısı sabit varsayılmıştır.

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