COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 20 A ANNÉE 1971

Variational Method and α–Starlike Functions

by

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Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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Variational Method and α-Starlike Functions (*)

Leman ÇELİKKANAT

Summary: In this paper α-starlike functions and meromorphic α-starlike functions are studied. Using Goluzin's variational method, variational formulas for these classes of functions are obtained, and some extremal problems have been solved. Also sharp bounds are obtained for α-starlike functions as:

$$r(1+r)^{-2(1-\alpha)} \leqq |f(z)| \leqq r(1-r)^{-2(1-\alpha)}, \quad (|z|=r<1)$$

and for meromorphic α-starlike functions as:

$$R(1-R^{-1})^{2(1-\alpha)} \leqq |F(\xi)| \leqq R(1+R^{-1})^{2(1-\alpha)}, \quad (|\xi|=R>1)$$

ξ1. *A representation formula for α-starlike functions*

α- starlike functions were introduced by M. S. Robertson [5], and then investigated by Ch. Pommerenke [4] in 1962.

Definition. A function

$$f(z) = z + a_2 z^2 + \dots \tag{1}$$

is called α- starlike if it is regular and schlicht in $|z|<1$, and there it satisfies the condition

$$\operatorname{Re}\left(\frac{z\, f'(z)}{f(z)}\right) > \alpha, \quad (0 \leqq \alpha < 1). \tag{2}$$

We shall denote the class of these functions by $S^*(\alpha)$. It is obvious that starlike functions, which map $|z|<1$ onto a star-

(*) This work has been presented as a Ph. D. thesis at the University of Ankara, Faculty of Science in January 1966.

like region with respect to the origin, will form the subclass $S^*(0)$ of $S^*(\alpha)$.

Teorem 1. Let

$$f(z) = z + a_2 z^2 + \dots$$

be a regular and schlicht function in $|z| < 1$. The necessary and sufficient condition for $f(z)$ to be α–starlike is the existance of integral representation

$$z\frac{f'(z)}{f(z)} = \alpha + (1-\alpha)\int_{-\pi}^{\pi}\frac{1+e^{-it}z}{1-e^{-it}z}\,d\gamma(t), \tag{3}$$

where $\gamma(t)$ is a nondecreasing function in $[-\pi,\pi)$, satisfying the condition $\gamma(\pi) - \gamma(-\pi) = 1$.

Proof. Let $f(z) \varepsilon S^*(\alpha)$, then a function $h(z)$ which is given by

$$h(z) = z^{-\alpha/(1-\alpha)}\, f(z)^{1/(1-\alpha)} \tag{4}$$

will be starlike. The logarithmic derivative of (4) yields

$$z\frac{h'(z)}{h(z)} = -\frac{\alpha}{1-\alpha} + \frac{1}{1-\alpha}\, z\frac{f'(z)}{f(z)}$$

and so

$$\mathrm{Re}\frac{zf'(z)}{f(z)} = -\frac{\alpha}{1-\alpha} + \frac{1}{1-\alpha}\,\mathrm{Re}\frac{zf'(z)}{f(z)} > 0. \tag{5}$$

Then by using Herglotz representation, we write

$$z\frac{h'(z)}{h(z)} = \int_{-\pi}^{\pi}\frac{1+e^{-it}z}{1-e^{-it}z}\,d\gamma(t)$$

and considering this in (5) we get (3).

Conversly, if $f(z)$ satisfies (3), by taking real parts of both sides we see that $\mathrm{Re}\,(z\frac{f'(z)}{f(z)}) > \alpha$, so $f(z)\,\varepsilon\, S^*(\alpha)$.

Dividing (3) by z, and integrating from zero to z, we get a representation formula for functions $f(z) \varepsilon S^*(\alpha)$ as :

$$f(z) = z \exp\left[-2(1-\alpha) \int_{-\pi}^{\pi} \log (1-e^{-it} z)\; d\gamma(t)\right] \tag{6}$$

where logarithm is understood as the branch vanishing at $z=0$.

§ 2. *Variational formulas for α–starlike functions*

By using Goluzin's variational method [1] we obtain variational formulas for α–starlike functions. Since this method is important we will refer to it briefly.

Let $q(z)$ be an analytic function which has a parametric representation as a Stieltjes integral

$$q(z) = \int_a^b p(z,t)\; d\gamma(t),$$

where a, b are given real numbers, $p(z,t)$ is a given function analytic in $|z| < 1$ for $a \leqq t \leqq b$, and $\gamma(t)$ runs through the set of all nondecreasing functions in $[a, b]$, under the condition

$$\int_a^b d\gamma(t) = \gamma(b) - \gamma(a) = 1.$$

For any two numbers t_1, t_2, $a \leqq t_1 < t_2 < b$, by changing $\gamma(t)$ in a suitable way in $t_1 < t < t_2$ and leaving unchanged outside of this interval, he has obtained the variational formula

$$q^*(z) = q(z) + \lambda \int_{t_1}^{t_2} p'_t(z,t)\; |\gamma(t)-c|\; dt \tag{7}$$

for $q(z)$, where λ is an arbitrary number in $[-1,1]$, and c is a certain constant independent of t and λ (but depends on the sign of λ). Next, assuming τ_1, τ_2, $a \leqq \tau_1 < \tau_2 < b$, be two jump points of the function $\gamma(t)$, for sufficiently small λ, he has obtained another variational formula for $q(z)$ as:

$$q^{**}(z) = q(z) + \lambda\; [\, p(z, \tau_1) - p(z, \tau_2)]. \tag{8}$$

Later, this method is improved by C. Uluçay. He gave a general formulation of the extremal function within the class E of analytic functions which considered by M. Goluzin, and he applied the result in a systematic way to analytic functions with positive real part and to typically-real functions [6].

If we denote the exponent in (6) by $\Psi(z)$ and apply formula (7), we obtain

$$\Psi^* (z) = \Psi(z) - 2\lambda(1-\alpha) \int_{t_1}^{t_2} \frac{i\, e^{-it} z}{1-e^{-it} z} \mid \gamma(t)-c \mid dt,$$

then denoting the corresponding function in the class $S^*(\alpha)$ by $f^*(z)$, and expanding this to a power series at $\lambda=0$, we get

$$f^*(z) = f(z) - 2\lambda(1-\alpha) \int_{t_1}^{t_2} f(z) \frac{ie^{-it}z}{1-e^{-it}z} \mid \gamma(t)-c \mid dt + 0(\lambda^2), \qquad (9)$$

(where $0(\lambda^2)$ is uniform with respect to z).

On the other hand by applying variational formula (8) to $\Psi(z)$ we find

$$\Psi^{**} (z) = \Psi (z) + 2\lambda(1-\alpha) \log \frac{1- e^{-i\tau_2} z}{1- e^{-i\tau_1} z} .$$

If we denote the corresponding function in the class $S^*(\alpha)$ by $f^{**}(z)$, for small values of λ we find

$$f^{**}(z) = f(z) + 2\lambda(1-\alpha) f(z) \log \frac{1- e^{-i\tau_2} z}{1- e^{-i\tau_1} z} + 0(\lambda^2) \qquad (10)$$

The formulas (9) and (10) are the two variational formulas for functions $f(z) \varepsilon S^* (\alpha)$.

In general, if $\gamma(t)$ is a step function with n jump points. $\tau_1 , \tau_2 , \dots , \tau_n$; $-\pi \leqq \tau_1 < \tau_2 < \dots < \tau_n < \pi$, and λ_k is its corresponding jump at τ_k , i.e.,

$$\lambda_k = \gamma(t_k+0) - \gamma(t_k-0), \qquad (\sum_{k=1}^{n} \lambda_k = 1, \lambda_k \geq 0)$$

then f(z) has the form

$$f(z) = \frac{z}{\sum_{k=1}^{n} (1 - e^{-i\tau_k} z)^{2(1-\alpha)\lambda_k}} \tag{11}$$

§ 3. *Solution of some extremal problems in the class* $S^*(\alpha)$

To solve some extremal problems in the class $S^*(\alpha)$ we shall use variational formulas which are obtained in the previous paragraph.

Theorem 2. For a given entire function $\varphi(w)$ and a given point z in $|z| < 1$ either of the functionals

$$\text{Re}\,[\,\varphi(\log \frac{f(z)}{z})\,] \quad \text{or} \quad |\,\varphi(\log \frac{f(z)}{z})\,| \tag{12}$$

attains its extremum in the class $S^*(\alpha)$ only for a function of the form

$$f(z) = \frac{z}{(1- e^{i\beta} z)^{2(1-\alpha)}} \,. \quad (\beta \text{ real})$$

Proof. Here we don't consider the case in which for the extremal function we have $\varphi'(\log \frac{f(z)}{z}) = 0$ (*).

The theorem asserts that , for every function $f(z) \varepsilon S^*(\alpha)$

$$\text{Re}\,[\,\varphi(\log \frac{f(z)}{z})] \leq \max_{\beta} \text{Re}\,[\varphi(\log \frac{1}{(1-e^{i\beta} z)^{2(1-\alpha)}})\,]$$

and

$$|\,\varphi(\log \frac{f(z)}{z})\,| \leq \max_{\beta} |\,\varphi(\log \frac{1}{(1- e^{i\beta} z)^{2(1-\alpha)}})\,| .$$

(*) Kirwan [2] has proved, in 1966, that this restriction can be removed by a suitable transformation.

Since $S^*(\alpha)$ is compact, there exists a solution of the problem and, it is enough to solve the problem only for one of the functionals (12). Because a function which gives maximum or minimum for $|\varphi(\log \frac{f(z)}{z})|$ also gives the same thing for $\mathrm{Re}[e^{i\eta}\varphi(\log \frac{f(z)}{z})]$ with a suitable chosen η, which is not different than the first functional of (12).

Denoting

$$I_f = \mathrm{Re}\ [\varphi(\log \frac{f(z)}{z})]$$

and f(z) being an extremal function, using variational formula(9) we get

$$\varphi(\log \frac{f^*(z)}{z}) = \varphi\,\{\log[\frac{f(z)}{z}(1-2\lambda(1-\alpha)\int_{t_1}^{t_2} \frac{i\,e^{-it}z}{1-e^{-it}z}|\gamma(t)-c|dt)+0(\lambda^2)]\}.$$

Expanding this to a power series at $\lambda=0$, and then taking real parts we get

$$I_{f*} = I_f - 2\lambda(1-\alpha)\,\mathrm{Re}\int_{t_1}^{t_2} \varphi'(\log \frac{f(z)}{z})\,\frac{ie^{-it}z}{1-e^{-it}z}\,|\gamma(t)-c|\,dt + 0(\lambda^2)$$

Since f(z) is an extremal function, the coefficient of λ must be zero, i.e.,

$$\int_{t_1}^{t_2} \mathrm{Re}\ [\varphi'(\log \frac{f(z)}{z})\,\frac{ie^{-it}z}{1-e^{-it}z}]\,|\gamma(t)-c|\,dt = 0\ .$$

This implies that: If

$$F(t) = \mathrm{Re}\ [\varphi'(\log \frac{f(z)}{z})\,\frac{ie^{-it}z}{1-e^{-it}z}] = 0\,, \qquad (13)$$

has no root in the interval (t_1, t_2), then along this interval $\gamma(t)-c$ must be zero, i. e., $\gamma(t)=c$ (constant). But if it has a solution, then $\gamma(t)$ may have discontinuities at the points t corresponding to the roots of (13). Since (13) is a quadratic equation with respect to e^{it}, then $\gamma(t)$ will be a step function with one or two jump points in $-\pi \leqq t < \pi$.

Now, assuming that $\gamma(t)$ has two jump points, say τ_1, τ_2; by using variational formula (10) we may write

$$\varphi(\log \frac{f^{**}(z)}{z}) = \varphi \{ \log [\frac{f(z)}{z}(1+2\lambda(1-\alpha) \log \frac{1-e^{-i\tau_2} z}{1-e^{-i\tau_1} z}) + 0(\lambda^2)] \}$$

Expanding this to a power series at $\lambda=0$, and taking real parts, we get

$$I_{f**} = I_f + 2\lambda(1-\alpha)\, \varphi'(\log \frac{f(z)}{z}) \log \frac{1-e^{-i\tau_2} z}{1-e^{-i\tau_1} z} + 0(\lambda^2).$$

Since f(z) is an extremal function, the coefficient of λ must be zero. This yields the condition that

$$\mathrm{Re}\, [\varphi'(\log \frac{f(z)}{z}) \log (1-e^{-it} z)]$$

has the same value at the points $t=\tau_1$, $t=\tau_2$. But in that case, by Rolle's theorem, its derivative with respect to t, which is F(t), would be zero at a cetain point τ_3 in the interval (τ_1, τ_2). Then the equation (13) would have more than two solutions in the interval $-\pi \leqq t < \pi$ which is impossible. This contradiction proves that $\gamma(t)$ must be a step function with only one jump point say $\beta \varepsilon [-\pi,\pi)$. Hence, by using formula (11) we see that, extremal function f(z) will have the form

$$f(z) = \frac{z}{(1-e^{i\beta} z)^{2(1-\alpha)}}. \quad (\beta \text{ real}) \qquad (14)$$

Application. Let us consider the functional

$$\varphi(w) = e^{aw} + b. \qquad (a, b \text{ constant})$$

By theorem 2, we know that the functional $|\varphi(\log \frac{f(z)}{z})|$ attains its maximum in the class $S^*(\alpha)$ only for a function of the form (14).

For $a = -\frac{1}{2(1-\alpha)}$, $b = -1$, we find

$$\left| e^{a\log(f(z)/z)} + b \right| = \left| \left(\frac{f(z)}{z}\right)^{-1/2(1-\alpha)} - 1 \right| \leqq \left| 1 - e^{i\beta} z - 1 \right| = |z| .$$

So, for any function $f(z) \varepsilon S^*(\alpha)$ we have

$$\left| \left(\frac{f(z)}{z}\right)^{1/2(1-\alpha)} - 1 \right| \leqq r \qquad (|z| = r < 1)$$

which yields the bounds

$$r(1+r)^{-2(1-\alpha)} \leqq |f(z)| \leqq r(1-r)^{-2(1-\alpha)} \quad (*).$$

Theorem 3. For a given entire function $\varphi(w)$ and a given point z in $|z| < 1$, either of the functionals

$$\operatorname{Re}\left[\varphi(\log f'(z))\right] \quad \text{or} \quad |\varphi(\log f'(z))| \tag{15}$$

attains its extremum in the class $S^*(\alpha)$ only for a function of the form

$$f(z) = \frac{z}{(1-e^{i\beta} z)^{\theta(1-\alpha)} (1-e^{i\eta} z)^{(2-\theta)(1-\alpha)}}, \tag{16}$$

where, $0 \leqq \theta < 2$, and β, η are real.

Proof. Here also we don't consider the case $\varphi'(\log f'(z)) = 0$, and by the same argument as in theorem 1, we shall prove this theorem only for the first functional of (15). Let

$$I_f = \operatorname{Re}\left[\varphi(\log f'(z))\right],$$

and assume that $f(z)$ is an extremal function. By using formula (9), we form $\varphi(\log f'^*(z))$, then expanding this to a power series at $\lambda = 0$, and taking real part, we get

(*) These bounds were obtained by M. S. Robertson [5] in a different way.

$$I_{f*} = I_f - 2\lambda(1-\alpha) \int_{t_1}^{t_2} \mathrm{Re} \left\{ \frac{\varphi'(\log f'(z))}{f'(z)} \frac{d}{dz} \frac{ie^{-it} z f(z)}{1-e^{-it} z} \right\} |\gamma(t)-c|\, dt + 0(\lambda^2). \quad (17)$$

The extremal property of f(z) implies that the coefficient of λ must be zero, that is.

$$\int_{t_1}^{t_2} \mathrm{Re} \left\{ \frac{\varphi'(\log f'(z))}{f'(z)} \frac{d}{dz} \frac{ie^{-it} z f(z)}{1-e^{-it} z} \right\} |\gamma(t)-c|\, dt = 0.$$

This implies that , if

$$F(t) = \mathrm{Re} \left\{ \frac{\varphi'(\log f'(z))}{f'(z)} \frac{d}{dz} \frac{ie^{-it} z f(z)}{1-e^{-it} z} \right\} = 0 \quad (18)$$

has no root in (t_1, t_2), then in this interval $\gamma(t)-c$ must be zero, i.e., $\gamma(t)=c$ (constant). If (18) has a solution in that interval, then $\gamma(t)$ may have discontinuities at the points t, corresponding to the roots of this equation, Since (18) is a fourth degree equation with respect to e^{it}, then $\gamma(t)$ will be a step function , with at most four jump points in $-\pi \leqq t < \pi$. Let us denote these points by τ_k $(k = 1, 2, 3, 4)$. Since $\gamma(t)$ is a step function, by using variational formula (10) we get

$$\varphi(\log f'^{**}(z)) = \varphi \left\{ \log \left[f'(z) \left(1-2\lambda(1-\alpha) \frac{1}{f'(z)} \frac{d}{dz} \frac{1-e^{-i\tau_{k+1}} z}{1-e^{-i\tau_k} z} + 0(\lambda^2)\right] \right\}, \quad (k=1, 2, 3).$$

Then expanding this to a power series at $\lambda=0$ we obtain

$$I_{f**} = I_f - 2\lambda (1-\alpha) \mathrm{Re} \left\{ \frac{\varphi'(\log f'(z))}{f'(z)} \frac{d}{dz} \left[f(z) \log \frac{1-e^{-i\tau_{k+1}} z}{1-e^{-i\tau_k} z} \right] \right\} + 0(\lambda^2).$$

The extremal property of f(z) implies that

$$\mathrm{Re} \left\{ \frac{\varphi'(\log f'(z))}{f'(z)} \frac{d}{dz} \left[f(z) \log \frac{1-e^{-i\tau_{k+1}} z}{1-e^{-i\tau_k} z} \right] \right\} = 0,$$

which means

$$\mathrm{Re} \left\{ \frac{\varphi'(\log f'(z))}{f'(z)} \frac{d}{dz} \left[f(z) \log (1-e^{-it} z) \right] \right\}$$

has the same value at each points of discontinuities. But in that case, its derivative with respect to t, which is F(t), would be zero at a certain point t in each interval (τ_{k+1}, τ_k). If $\gamma(t)$ has more than two jump points, the number of roots of (18) would exceed four, which is impossible, Hence we conclude that $\gamma(t)$ must be a step function with only two jump points, say β and η. Then by formula (11), f(z) has the form (18).

Theorem 4. For a given entire function $\varphi(w)$ and a given point z in $|z|<1$, either of the functionals

$$\mathrm{Re}\left[\varphi\left(\log \frac{z^k f'(z)}{f(z)^k}\right)\right] \quad \text{or} \quad \left|\varphi\left(\log \frac{z^k f'(z)}{f(z)^k}\right)\right|.$$

attains its extremum in the class $S^*(\alpha)$ only for a function of the form (16)

Proof. Here also we neglect the case for which $\varphi(\log f'(z)) = 0$. It is sufficient to investigate only the functional

$$I_f = \mathrm{Re}\left[\varphi\left(\log \frac{z^k f'(z)}{f(z)^k}\right)\right]$$

By using the variational formula (9) we get

$$\varphi\left(\log \frac{z^k f'^*(z)}{f(z)^k}\right) = \varphi\left\{\log\left[\frac{z^k f'(z)}{f(z)^k}\left(1-2\lambda(1-\alpha)\frac{1}{f'(z)}\frac{d}{dz}\frac{ie^{-it} zf(z)}{1-e^{-it} z}\cdot |\gamma(t)-c|\, dt\right) + 0(\lambda^2)\right]\right\}$$

for small values of λ, the real part of this is

$$I_{f*} = I_f - 2\lambda(1-\alpha)\int_{t_1}^{t_2} \mathrm{Re}\left\{\frac{\varphi\left(\log \frac{z^k f'(z)}{f(z)^k}\right)}{f'(z)}\frac{d}{dz}\frac{ie^{-it} zf(z)}{1-e^{-it} z}\right\}|\gamma(t)-c|\, dt + 0(\lambda^2). \qquad (19)$$

The only difference between (19) and (17) is the appearence of the factor $\varphi'\left(\log \frac{z^k f'(z)}{f(z)^k}\right)$ instead of $\varphi'(\log f'(z))$, and since we exclude from consideration the case for which $\varphi'\left(\log \frac{z^k f'(z)}{f(z)^k}\right) = 0$

and $\varphi'(\log f'(z)) = 0$, then the same result remains true also for these functionals.

§ 5. *Meromorphic α–starlike functions*

These functions are introduced by Ch. Pommerenke [3] in 1962. In this paragraph we shall form the variational formulas for meromorphic α–starlike functions, then using these formulas we shall obtain some sharp bounds for these functions.

Definition . Let

$$W = F(\xi) = \xi + b_0 + b_1 \xi^{-1} + \dots$$

be an analytic and schlicht function in $|<|\xi|<\infty$, $F(\xi)$ is called meromorphic α–starlike if for every ξ in $1<|\xi|<\infty$

$$\mathrm{Re}\,(\xi \frac{F'(\xi)}{F(\xi)}) \geq \alpha \qquad (0 \leqq \alpha < 1)$$

is satisfied.

We shall denote the class of these functions by $S(\alpha)$. It is obvious that the meromorphic starlike functions form the subclass $S(0)$ of $S(\alpha)$

Theorem 5. Let

$$F(\xi) = \xi + b_0 + b_1 \xi^{-1} + \dots$$

be analytic and schlicht in $1<|\xi|<\infty$. The necessary and sufficient condition for $F(\xi)$ to be meromorphic α–starlike is the existance of integral representation

$$\xi \frac{F'(\xi)}{F(\xi)} = \alpha + (1-\alpha) \int_{-\pi}^{\pi} \frac{1+e^{it}\xi^{-1}}{1-e^{it}\xi^{-1}}\, d\gamma(t). \tag{20}$$

Where $\gamma(t)$ is a nondecreasing function in $[-\pi, \pi)$, subject to the condition $\gamma(\pi) - \gamma(-\pi) = 1$.

Proof. Condition is necessary: If $F(\xi) \varepsilon S(\alpha)$, a function $H(\xi)$ which is defined by

$$H(\xi) = \left(\frac{F(\xi)}{\xi^{\alpha}} \right)^{1/(1-\alpha)} \tag{21}$$

is meromorphic starlike. Since the logaritmic derivative of (21) gives

$$\xi \frac{H'(\xi)}{H(\xi)} = - \frac{\alpha}{1-\alpha} + \frac{1}{1-\alpha} \xi \frac{F'(\xi)}{F(\xi)} \tag{22}$$

which shows that

$$\text{Re} \left(\xi \frac{H'(\xi)}{H(\xi)} \right) \geq 0.$$

Hence we may write

$$\xi \frac{H'(\xi)}{H(\xi)} = \int_{-\pi}^{\pi} \frac{1+e^{it}\xi^{-1}}{1-e^{it}\xi^{-1}} d\gamma(t) \tag{23}$$

and using this in (22) we get (20).

Condition is sufficient: Since real part of the last term in (20) is not negative, then $\text{Re} \left(\xi \frac{F'(\xi)}{F(\xi)} \right) \geq \alpha$. i.e., $F(\xi) \varepsilon S(\alpha)$.

Dividing (23) by ξ and integrating it from zero to ξ we obtain the representation formula

$$H(\xi) = \xi e^{2\int_{-\pi}^{\pi} \log(1-e^{it}\xi^{-1})\, d\gamma(t)} \tag{24}$$

for meromorphic starlike functions . By replacing (24) in (21) we get a representation formula for meromorphic α–starlike functions as:

$$F(\xi) = \xi e^{2(1-\alpha)\int_{-\pi}^{\pi} \log(1-e^{it}\xi^{-1})\, d\gamma(t)} \tag{25}$$

Now, by the use of Goluzin's variational method we obtain two variational formulas for meromorphic α–starlike functions, then by using these formulas we shall solve some extremal problems in the class of these functions and obtain some sharp bounds.

Let E_G denote the class of meromorphic functions represented by a Stieltjes integral

$$Q(\xi) = \int_a^b G(\xi, t)\, d\gamma(t),$$

where a, b are given real numbers, $G(\xi, t)$ is a given function analytic in $1<|\xi|<\infty$, for $a \leqq t \leqq b$, and $\gamma(t)$ is any nondecreasing function in $[a, b]$ satisfying $\gamma(b) - \gamma(a) = 1$. By the same way as of § 1, we get variational formulas

$$Q^*(\xi) = Q(\xi) + \lambda \int_{t_1}^{t_2} G'_t(\xi, t) \; | \gamma(t) - c | \; dt \tag{26}$$

and

$$Q^{**}(\xi) = Q(\xi) + \lambda [G(\xi, \tau_1) - G(\xi, \tau_2)] \tag{27}$$

for functions $Q(\xi) \varepsilon E_G$.

Writing (25) as $F(\xi) = \xi\, e^{\Psi(\xi)}$ and applying variational formula (26) to this exponent we get

$$\Psi^*(\xi) = \Psi(\xi) - 2\lambda(1-\alpha) \int_{t_1}^{t_2} \frac{i\, e^{it}\, \xi^{-1}}{1 - e^{it}\, \xi^{-1}} \; | \gamma(t) - c | \; dt.$$

If we denote the corresponding function in $S(\alpha)$ by $F^*(\xi)$, and expand this to a pover series at $\lambda=0$ we get

$$F^*(\xi) = F(\xi) - 2\lambda(1-\alpha) \int_{t_1}^{t_2} F(\xi) \frac{i\, e^{it}\, \xi^{-1}}{1 - e^{it}\, \xi^{-1}} | \gamma(t) - c | \; dt + 0(\lambda^2). \tag{28}$$

If τ_1 and τ_2, $-\pi \leqq \tau_1 < \tau_2 < \pi$, are two jump points of $\gamma(t)$, applying formula (27) to $\Psi(\xi)$ we get $\Psi^{**}(\xi)$. Then expanding the expression

$$\begin{aligned} F^{**}(\xi) &= \xi \exp [\Psi^{**}(\xi)] \\ &= \xi \exp \left[\Psi(\xi) + 2\lambda \; (1-\alpha) \log \frac{1 - e^{i\tau_1} \xi^{-1}}{1 - e^{i\tau_2} \xi^{-1}} \right] \end{aligned}$$

to a power series at $\lambda=0$ we get $F^{**}(\xi)$ as:

$$F^{**}(\xi) = F(\xi) + 2\lambda(1-\alpha)\, F(\xi)\, \log \frac{1-e^{i\tau_1}\xi^{-1}}{1-e^{i\tau_2}\xi^{-1}} + 0(\lambda^2). \tag{29}$$

In general, if $\gamma(t)$ is a step function with n jump points τ_1, $\tau_2, \ldots, \tau_n$, $-\pi \leqq \tau_1 < \tau_2 < \ldots < \tau_n < \pi$, and λ_k is its jump at the point τ_k, i.e., $\lambda_k = \gamma(\tau_k+0) - \gamma(\tau_k-0)$, then it is easy to see that $F(\xi)$ will have the form

$$F(\xi) = \xi \sum_{k=1}^{n} (1-e^{i\tau_k}\xi^{-1})^{2(1-\alpha)\lambda_k} \qquad (\lambda_k \geq 0\,,\ \sum_{k=1}^{n} \lambda_k = 1)\,.$$

§ 6. *Solutions of some extremal problems in the class* S(α).

The similar teorems to 2-4 are easily proved for functions $F(\xi) \varepsilon S(\alpha)$.

Theorem 6. For a given entire function $\Phi(W)$ and a given point ξ in $1 < |\xi| < \infty$, either of the functionals

$$\mathrm{Re}\,[\,\Phi\,(\log \frac{F(\xi)}{\xi})] \qquad \text{or} \qquad |\,\Phi\,(\log \frac{F(\xi)}{\xi})]\,|$$

attains its extremum in the class $S(\alpha)$ only for a function of the form

$$F(\xi) = \xi(1-e^{i\beta}\xi^{-1})^{2(1-\alpha)}$$

Proof. Here we also neglect the case in which $\Phi'(\log \frac{F(\xi)}{\xi}) = 0$ for extremal function. Denote by J_F :

$$J_F = \mathrm{Re}[\,\Phi\,(\log \frac{F(\xi)}{\xi})]$$

and assume that $F(\xi)$ is an extremal function, using the variational formula (28) and following the same proceedure as in the proof of theorem 2, we get

$$J_{F*} = J_F - 2\lambda(1-\alpha)\mathrm{Re}\int_{t_1}^{t_2} \Phi'(\log \frac{F(\xi)}{\xi})\, \frac{i\, e^{it}\xi^{-1}}{1-e^{it}\xi^{-1}}\, |\gamma(t)-c|\, dt + 0(\lambda^2).$$

The extremal property of $F(\xi)$ implies that $\gamma(t)$ is a step function which may have discontinuities only at the points t corresponding to the roots of

$$\mathrm{Re}\ [\ \Phi'(\log \frac{F(\xi)}{\xi})\ \frac{i\,e^{it}\,\xi^{-1}}{1-e^{it}\,\xi^{-1}}\ ] = 0. \tag{30}$$

Since equation (30) is a quadratic equation with respect to e^{it}, $\gamma(t)$ may have at most two jump points, say τ_1, τ_2, in (30), $-\pi \leqq t < \pi$. In that case by using variatonal formula (29), for small values of λ we get

$$J_{F**} = J_F + 2\lambda(1-\alpha)\mathrm{Re}\,[\Phi'(\log \frac{F(\xi))}{\xi}\ \log \frac{1-e^{i\tau_2}\,\xi^{-1}}{1-e^{i\tau_1}\,\xi^{-1}}] + 0(\lambda^2).$$

Since $F(\xi)$ is an extremal function.

$$\mathrm{Re}\ [\ \Phi'(\log \frac{F(\xi)}{\xi})\ \log\ (1-e^{it}\,\xi^{-1})] \tag{31}$$

must have the same value at the points $t=\tau_1$, $t=\tau_2$. But in that case, the derivative of (31) with respect to t would be zero at a certain point τ_3 in the interval (τ_1, τ_2), so the number of roots of (30) would be more than two, which is impossible. Hence $\gamma(t)$ is a step function with only one jump point, say τ, in $-\pi \leqq t < \pi$. This implies that $F(\xi)$ has the form

$$F(\xi) = \xi(1-e^{i\tau}\xi^{-1})^{2(1-\alpha)}\ .$$

Application. Let us consider the functional

$$\Phi(W) = e^{aw} + b\ . \qquad \text{(a, b constant)}$$

By theorem 6, we know that the functional $|\,\Phi(\log \frac{F(\xi)}{\xi})\,|$ attains its extremum in the class $S(\alpha)$ only for a function of the form

$$F(\xi) = \xi\,(1-e^{i\tau}\xi^{-1})^{2(1-\alpha)}$$

Let $|\,\xi\,| = r$, for $a = \frac{1}{2(1-\alpha)}$ and $b = -1$, we get

$$|\Phi(\log \frac{F(\xi)}{\xi})| = |(\frac{F(\xi)}{\xi})^{\frac{1}{2(1-\alpha)}} - 1| \leqq |1-e^{i\tau}\xi^{-1} - 1| = R^{-1}$$

So, for any function $F(\xi) \varepsilon S(\alpha)$ we have the bounds

$$R(1-R^{-1})^{2(1-\alpha)} \leqq |F(\xi)| \leqq R(1+R^{-1})^{2(1-\alpha)}$$

These bounds have also been found by Ch . Pommerenke [3] in a different way.

Finally we shall state two theorems but without giving their proof, since they are similar to the theorems 3 and 4.

Theorem 7. For a given entire function $\Phi(W)$ and a given point ξ in $1 < |\xi| < \infty$, either of the functionals

$$\text{Re } [\Phi(\log F'(\xi))] \quad \text{or} \quad |\Phi(\log F'(\xi))|$$

attains its extremum in the class $S(\alpha)$ only for a function of the form

$$F(\xi) = \xi(1-e^{i\beta}\xi^{-1})^{\theta(1-\alpha)}(1-e^{i\eta}\xi^{-1})^{(2-\theta)(1-\alpha)} \tag{32}$$

where $0 \leqq \theta < 2$, and β , η are real numbers.

Theorem 8. For a given entire function $\Phi(W)$ and a given point ξ in $1<|\xi|<\infty$, either of the functionals

$$\text{Re } [\Phi(\log \frac{\xi^k F'(\xi)}{F(\xi)^k})] \quad \text{or} \quad |\Phi(\log \frac{\xi^k F'(\xi)}{F(\xi)^k}|$$

attains its extremum in the class $S(\alpha)$ only for a function of the form (32).

I would like to express my grateful thanks to Prof. C. Uluçay for his help and valuable advice on preperation of this paper.

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ÖZET

Bu çalışmada α–yıldızıl fonksiyonlarla meromorfik α–yıldızıl fonksiyonlar incelenmiştir. Goluzin'nin varyasyon metodu kullanılarak bu sınıflardaki fonksiyonlar için varyasyon formülleri elde edilmiş ve bazı ekstremal problemler çözülmüştür.

Ayrıca α–yıldızıl fonksiyonlar için

$$r(1+r)^{-2(1-\alpha)} \leqq |f(z)| \leqq r(1-r)^{-2(1-\alpha)}, \quad (|z| = r < 1);$$

meromorfik α–yıldızıl fonksiyonlar için ise

$$R(1-R^{-1})^{2(1-\alpha)} \leqq |F(\xi)| \leqq R(1+R^{-1})^{2(1-\alpha)}, \quad (|\xi| = R > 1)$$

kesin sınırları elde edilmiştir.

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Ankara Üniversitesi Basımevi, 1972