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Série A: Mathématiques, Physique et Astronomie

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Faculté Des Sciences de l'Université D'Ankara
Ankara, Turquie

ÖNSÖZ

Cumhuriyetimizin kuruluşunun 50. yıl dönümünde, Ankara Üniversitesi Fen Fakültesinin Dekanı olarak bulunmam benim için çok mutlu bir raslantıdır.

Yarım yüzyıl önce büyük bir azim ile milletini bir bütün halinde toplayan büyük kurtarıcımız Atatürk'ün aziz hatırası önünde memleketimizin bütün bilim adamlarının saygıyla eğildiğine eminim. İnsanlık haysiyet ve özgürlüğüne göz diken, kıskanç ve hırslı ruhların aldatmaya çalıştığı gençlerimiz, ümit ederimki, Atatürk'ün kendilerine emanet ettiği bu Cumhuriyet'in, içine atılmaya çalışıldığı durumu artık anlamışlar ve ancak Atatürk'ün kendilerine gösterdiği "hayatta en hakiki mürşit olan ilim" yolunda çalışarak memleketlerine hizmet edebileceklerinin bilincine varmışlardır.

Gençlerimizin bu teknik ve uzay çağında bir Fen Fakültesi öğrencisi olmanın değerini çok iyi takdir ettiklerine ve fedakâr milletimizin kendilerine sağladığı bu imkânı en iyi şekilde kullanmaya azimli olduklarına inanmaktayım. Fakültemizde son yıllarda % 80'e çıkan başarı durumu da bunun delilidir.

Öğrencilerimizin ve Fakültemiz mensuplarının Türkiyenin geleceğine katkıda bulunmak için daha kuvvetle çalışacaklarına bu yıl dönümünde bir kere daha söz veriyoruz.

DEKAN
Prof. Dr. Sevinç KAROL

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Proof of Bieberbach's Conjecture

by

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

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Adres: Fen Fakültesi Tebliğler Dergisi Fen Fakültesi, Ankara, Turquie.

Proof of Bieberbach's Conjecture [1)]

C. ULUÇAY

De La Faculté Des Sciences De L'Université D'Ankara

SUMMARY

It is shown by the method of 2-dimensional cross-section that for the class S of analytic and schlicht functions

$$f(z) = z + a_2 z^2 + \ldots, \quad |z| < 1,$$

the inequality

$$|a_n| \leqq n$$

is always true, with equality for any n, $n \geqq 2$ if and only if $f(z)$ is a Koebe function.

Survey. In this paper we prove the famous Bieberbach's conjecture, i. e., for the class S of analytic and schlicht functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, \quad |z| < 1,$$

the inequality

$$|a_n| \leqq n$$

is always true, with equality for any n, $n \geqq 2$, if and only if $f(z)$ is a Koebe function

$$\frac{z}{(1-e^{i\theta}z)^2} = z + 2e^{i\theta} z^2 + 3e^{2i\theta} z^3 + \ldots, \quad \theta \text{ real.}$$

Up to now, the conjecture has only been proved for $n = 2, 3, 4$ (see: [2], [3], [4], [5]).

As usual let V_{n-1} be the set of points

$$\tilde{a} = (a_2, a_3, \ldots, a_{n-1})$$

belonging to functions

1) This work is dedicated to the 50^{th} anniversary of the Turkish Republic.

$$f(z) = z + a_2 z^2 + \dots + a_{n-1} z^{n-1} + a_n z^n + \dots$$

of class S. Let $\dot{\sigma}_n = \sup |a_n|$; evidently $\dot{\sigma}_n \geqq n$, and it will suffice to consider only the class $\dot{S}_n$ of so-called extremal functions

$$\sigma(z) = z + \sigma_2 z^2 + \dots + \dot{\sigma}_n z^n + \dots$$

in S with respect to the n-th coefficient. In the sequel the dot will always refer to such a coefficient for which $\sigma(z)$ is extremal.

The main idea, from which the Bieberbach's conjecture (Theorem II) is easily derived, is formulated in Theorem I. This idea, i. e., any extremal function $\sigma(z)$ with respect to the n-th coefficient implies that the point $(\sigma_2, \sigma_3, \dots, \sigma_{n-1})$ should be a boundary point of V_{n-1}, is quite intuitive. For, let us associate to each point $\widetilde{a} \varepsilon V_{n-1}$ the number $t_n = t_n(\widetilde{a})$ uniquely defined by the 2- dimensional cross-section π of V_n obtained by holding $a_2, \dots, a_{n-1}$ fixed and letting a_n vary, and such that

$$t = (a_2, a_3, \dots, a_{n-1}, \tau_n)$$

is a boundary point of V_n, lying on π, in which

$$t_n = \operatorname{Re}\tau_n = \max \operatorname{Re} a_n$$

It is then natural to expect that max t_n must occur at some point $\widetilde{p}$ on the boundary of V_{n-1}. It should be noticed that the above idea suggets at the same time the method of proof which may be called the method of 2- dimensional cross-section. This, in turn involves a certain important inequality due to Teichmüller ([1], p. 105). Let

$$a = (a_2, \dots, a_{n-1}, a_n)$$

be a boundary point of V_n. It is known that a or what is the same thing $w = f(z)$ satisfies a differential equation of the form

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z)$$

where

$$P(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{w^\nu}, \qquad Q(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu}.$$

If $A_{n-1} \neq 0$, then

$$\mathrm{Re}\ \{(b_n - a_n)\ A_{n-1}\} \leqq 0,$$

where $b = (a_2, a_3, ..., a_{n-1}, b_n)$ is any point of V_n in π. The equality holds if and only if $a_n = b_n$. i. e., $a = b$. From this inequality it follows that π as well as the set of interior points of V_n belonging to π is convex. We shall call the set of interior points of V_n belonging to π the interior of π. Due to its importance, the above inequality will be called by us the *Teichmüller's Principle.*

Introduction. To make the paper self-contained, we recall once more some known facts about the n- th coefficient region V_n in $(2n—2)$ - dimensional real Euclidean space whose points $(a_2, a_3, ..., a_n)$ correspond to functions of class S. For details, the reader is referred to [1], Chapter 1. The topological structure of V_n is almost evident. First of all, V_n is bounded and closed since $| a_n | < en$ and S is compact. Moreover the function $f(z) = z$ being in S and bounded it readily follows that the origin is an interior point of V_n. Finally, it can be shown that V_n is connected and topologically equivalent to the closed $(2n—2)$ - dimensional full sphere. For example, the coefficient-region V_2 of points (a_2) is simply the disc $| a_2 | \leq 2$. For, any function $f(z) \varepsilon S$ such that $|a_2| < 2$, a_2 is an interior point of V_2, and to each boundary point $a_2 = 2e^{i\theta}$ corresponds a unique function in S, i.e., $f(z) = \dfrac{z}{(1-e^{i\theta}\ z)^2}$. It is convenient to introduce at this moment the following terminology. We say that the point $(a_2, a_3, ..., a_n)$ belongs to a function

$$f(z) = z + b_2 z^2 + ... + b_n z^n + ...$$

of class S and that $f(z)$ belongs to the point $(a_2, a_3, ..., a_n)$ if

$$a_\nu = b_\nu, \quad \nu = 2, 3, ..., n$$

If $a = (a_2, a_3, ..., a_n)$ belongs to $f(z)$ and is an interior point of V_n then there is an $\varepsilon > 0$ such that all points $c = (c_2, c_3, ..., c_n)$ satisfying the inequality

$$|| a - c || = \left(\sum_{\nu=2}^{n} | c_\nu - a_\nu |^2 \right)^{\frac{1}{2}} < \varepsilon$$

are interior points of V_n, and so there is at least one function of class S which belongs to $(c_2, c_3, \ldots, c_n)$. In particular, the point $(\rho a_2, \rho^2 a_3, \ldots, \rho^{n-1} a_n)$ is an interior point of V_n for some $\rho > 1$. It follows readily that there is a bounded function of class S which belongs to $(a_2, a_3, \ldots, a_n)$. Conversely, if $f(z)$ is a bounded function of class S and belongs to $(a_2, \ldots, a_n)$ then the latter is an interior point of V_n. The boundary and interior points of V_n can be characterized as follows : If $(a_2, a_3, \ldots, a_n)$ is a boundary point of V_n, then there is only one function of class S belonging to it, whereas if it is an interior point of V_n then there is more than one function of class S belonging to it.

Lemma I. (i) Let $p = (\sigma_2, \ldots, \sigma_{n-1}, \dot{\sigma}_n)$. Then p satisfies a differential equation ϑ_n of the form

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P(w) = Q(z), \quad |z| < 1, \; w = \sigma(z),$$

where

$$P(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{w^\nu}, \qquad Q(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu},$$

$$B_{-\nu} = \overline{B}_\nu, \; Q(e^{i\theta}) \geqq 0 \text{ and } A_{n-1} = B_{n-1} = 1.$$

Here $Q(z)$ has on $|z| = 1$, at least one zero, which must be of even order.

(ii) Let p_0 be a boundary point of V_n, near p. Then p_0 satisfies a differential equation ϑ_n^0 of the same type

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P_0(w) = Q_0(z), \quad |z| < 1$$

where

$$P_0(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu^0}{w^\nu}, \qquad Q_0(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu^0}{z^\nu},$$

$$B^0_{-\nu} = \overline{B}_\nu^0, \qquad Q_0(e^{i\theta}) \geqq 0$$

and $A^0_{n-1} \to 1$ as $p_0 \to p$ through boundary points.

Proof. To show the first part of the lemma, we use the Schaeffer Spencer variational formula [1]. We see that p satisfies the differential equation ϑ_n of the form

$$\frac{z^2\sigma'(z)^2}{\sigma(z)^3} S_n\left(\frac{1}{\sigma(z)}\right) = (n-1)\dot{\sigma}_n + \sum_{\nu=1}^{n-1} \frac{\nu\sigma_\nu}{z^{n-\nu}} + \sum_{\nu=1}^{n-1} \overline{\nu\sigma_\nu} z^{n-\nu}$$

where S_n is defined by

$$\frac{\sigma(z)^2}{1-\dfrac{\sigma(z)}{\sigma(z_0)}} = \sum_{n=2}^{\infty} S_n\left(\frac{1}{\sigma(z_0)}\right) z^n.$$

ϑ_n is of the required type. We see immediately that $B_{n-1} = 1$, while an easy calculation shows that $A_{n-1} = 1$. Thus $A_{n-1} = B_{n-1}$ as it should be.

As to the second part of the lemma, let p_0 be a boundary point of V_n. We know from the general theory of the coefficients of schlicht functions ([1], pp. 36–43), that p_0 satisfies a differential equation $\vartheta_n{}^0$ of the form described in the lemma, i. e.,

$$\left(\frac{z}{w}\frac{dw}{dz}\right)^2 P_0(w) = Q_0(z), \quad |z| < 1$$

where

$$P_0(w) = \sum_{\nu=1}^{n-1} \frac{A_\nu{}^0}{w^\nu}.$$

The boundary $B^0{}_w$ in the w- plane corresponding to $|z| = 1$ in the mapping σ_0 (z) belonging to p_0 consists of loci defined by

$$\text{Re} \int (P_0(w))^{\frac{1}{2}} \frac{dw}{w} = \text{constant}.$$

If A^0 designates the $(n-1)$ - tuple $(A_1{}^0, A_2{}^0, \ldots, A^0{}_{n-1})$ then $B^0{}_w = B^0{}_w(A^0)$ is a function of A^0 as p_0 varies on the boundary of V_n. Let

$$A = (A_1, A_2, \ldots, A_{n-1})$$

be the vector associated with ϑ_n. In view of the extremal property of p, $B_w = B_w(A)$ is a single analytic arc extending to infinity, without critical points and of mapping radius unity. It is known then that $B_w(A)$ is a continuous function of A (cf. pp. 44-87, Lemma XXII). Finally, it follows from a known argument (loc. cit. pp. 40-41 and p. 111-112) that there is a one to one continuous correspondence between the boundary points of V_n in the neighborhood of p and a set of vectors containing A. Hence if p_0 is sufficiently near p it follows from the foregoing continuity argument that p_0 satifies a differential equation ϑ_n^0 in which A^0 is arbitrarily close to A, i. e.

$$||A^0 - A|| = \left(\sum_{\nu=1}^{n-1} | A_\nu^0 - A_\nu |^2 \right)^{\frac{1}{2}} \to 0 \text{ as } p_0 \to p.$$

In particular $A^0_{n-1} \to A_{n-1}$ as $p_0 \to p$ through boundary points.

Theorem I. Let $\sigma(z) \varepsilon \dot{S}_n$. Then $(\sigma_2, \sigma_3, ..., \sigma_{n-1})$ is a boundary point of V_{n-1}.

Proof. Consider the 2- dimensional cross-section of V_n obtained by holding $\sigma_2, ..., \sigma_{n-1}$ fixed and varying the last coordinate in

$$p = (\sigma_2, \sigma_3, ..., \sigma_{n-1}, \dot{\sigma}_n).$$

Suppose on the contrary that $(\sigma_2, \sigma_3, ..., \sigma_{n-1})$ is an interior point of V_{n-1}. Then the following properties hold [6]:

Property I. let $p = (\sigma_2, \sigma_3, ..., \sigma_{n-1}, \dot{\sigma}_n)$. Suppose that $b = (\sigma_2, \sigma_3, ..., \sigma_{n-1}, b_n)$ is an interior point of V_n. Then each point of the segment bp, save p, is an interior point of V_n.

Indeed , let π denote the 2- dimensional cross-section of V_n obtained by holding $\sigma_2, ..., \sigma_{n-1}$ fixed. Owing to the fact that π is convex, the segment bp lies in π. Suppose that $r \neq p$ is the first boundary point of V_n on the line segment bp. Since the interior of π is also convex it follows that every point on rp is a boundary point of V_n. Let p_0 be any boundary point of V_n lying on rp and sufficiently near p. Applying Teichmüller's Principle to p_0, we have

$$\text{(1)} \qquad \text{Re } \{ (\dot{\sigma}_n - \tau^0_n) A^0_{n-1} \} < 0, \; \dot{\sigma}_n \neq \tau^0_n,$$

where τ^0_n is the last coordinate in $p_0 = (\sigma_2, \sigma_3, ..., \sigma_{n-1}, \tau^0_n)$. By Lemma I, (ii), $A^0_{n-1} \to 1$ as $p_0 \to p$. Recalling that τ^0_n lies on the segment $b_n \dot{\sigma}_n$ which is fixed and non perpendicular to the real axis since b_n lies in the interior of the disc G, centre at the origin and radius $\dot{\sigma}_n$, it is readily seen that for p_0 sufficiently near p, this inequality is impossible. One can also see this by calculation. In fact, (1) can be written as

$$\text{(2)} \qquad \dot{\sigma}_n - u^0_n < - v^0_n \tan \theta$$

where $\tau^0_n = u^0_n + iv^0_n$, $\arg A^0_{n-1} = \theta$ with $\theta \to 0$ as $p_0 \to p$. Setting $| \tau^0_n - \dot{\sigma}_n | = \varepsilon$ and assuming b_n not real, then $\dot{\sigma}_n - u^0_n$ and v^0_n are of precise order ε. Hence (2) is impossible even if $\theta < 0$, for the right hand side of (2) is of higher order than ε. This contradiction implies $r = p$.

Property II. Let $p = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \dot{\sigma}_n)$ Suppose that $b = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, b_n)$ is an interior point of V_n. Let $\widetilde{p} = (\sigma_2, \ldots, \sigma_{n-1}, \widetilde{\sigma}_n)$ be a boundary point of V_n sufficiently near p. Then each point of $\widetilde{bp}$, save $\widetilde{p}$ is an interior point of V_n.

Indeed, let $\widetilde{r}$, $\widetilde{p}_0$ and $\widetilde{\tau}^0_n$ be as before. Applying Teichmüller's Principle to $\widetilde{p}_0$, we obtain

$$\text{(3)} \qquad \operatorname{Re} \{(\widetilde{\sigma}_n - \widetilde{\tau}^0_n)\, \widetilde{A}^0_{n-1}\} < 0, \widetilde{\sigma}_n \neq \widetilde{\tau}^0_n.$$

As in property I, the result is geometrically evident. For, by assumption

$$\operatorname{Re} \widetilde{\tau}^0_n < \operatorname{Re} \widetilde{\sigma}_n,$$

and the line segment $\widetilde{\tau}^0_n \widetilde{\sigma}_n$ lying on $b_n \widetilde{\sigma}_n$ is never perpendicular to the real axis as $\widetilde{\sigma}_n \to \dot{\sigma}_n$ The latter property and thereby the sense of the inequality in $\operatorname{Re} \widetilde{\tau}^0_n - \operatorname{Re} \widetilde{\sigma}_n < 0$ will be preserved after application of the infinitesimal rotation $\operatorname{Arg} \widetilde{A}^0_{n-1} = \theta$ to $\widetilde{\tau}^0_n$ and $\widetilde{\sigma}_n$ respectively. Namely,

$$\text{(4)} \qquad \operatorname{Re} \widetilde{\tau}^0_n e^{i\theta} - \operatorname{Re} \widetilde{\sigma}_n e^{i\theta} = \operatorname{Re} \{(\widetilde{\tau}^0_n - \widetilde{\sigma}_n) e^{i\theta}\} < 0.$$

However (4) contradicts (3) and the assertion follows.

We also conclude that if $\widetilde{p}$ is sufficently near p, then $\widetilde{p}$, is the only boundary point on $b\,\widetilde{p}$.

Let us consider the 2- dimensional cross-section π. It is clear that as the point

$$a = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, a_n). \quad |a_n| \leqq \dot{\sigma}_n$$

describes π, the last coordinate a_n will describe in the complex plane a set of points Π lying in the disc G, centre at the origin and radius $\dot{\sigma}_n$, and which is convex. We shall say that a_n is the projection of a and Π is the projection of π. $a_n(a)$ is said to be an interior or a boundary point of Π (π) if a is an interior or a boundary point of V_n respectively. We recall that the interior of Π as well as the interior of π is convex. In view of property I and property II the set γ of boundary points of π containing p and sufficiently near p, is a continuous arc containing p. Let Γ be the projection of γ in Π. Γ passes through $\dot{\sigma}_n$, and a_n which is the projection of $a \varepsilon \gamma$, describes Γ. It follows that every point on the line segment $b_n\, a_n$, save a_n is an interior point of Π, and that Γ is a continuous arc.

Let us show that Γ can have no point in common with the circumference of G, save $\dot{\sigma}_n$, as a_n approaches $\dot{\sigma}_n$. If $a_n = \dot{\sigma}_n e^{i\varphi}$ is such a point, then the point

$$p_\varphi = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \dot{\sigma}_n e^{i\varphi})$$

would lie on the boundary of π. Using the notation of Lemma I, it is easily seen that B_0 has the representation

$$B_0 = -\min_{|z|=1} \left\{ \sum_{\nu=1}^{n-1} \left(\frac{B_\nu}{z^\nu} + \overline{B}_\nu z^\nu \right) \right\},$$

where $B_0 = (n-1)\,\dot{\sigma}_n$, $B_\nu = (n-\nu)\,\sigma_{n-\nu}$. Through rotation $-\varphi\,/\,(n-1)$, the point p_φ takes the form

$$p_{-\varphi/(n-1)} = (\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1}, \dot{\sigma}_n)$$

and

$$B_0 = -\min_{|z|=1} \left\{ \sum_{\nu=1}^{n-1} \left(\frac{B_\nu^*}{z^\nu} + \overline{B}_\nu^*\, z^\nu \right) \right\}, \tag{5}$$

where

$$B_\nu^* = (n-\nu)\,\sigma^*_{n-\nu}, \quad \sigma^*_{n-\nu} = \sigma_{n-\nu}\, e^{-i\frac{(n-\nu-1)}{n-1}\varphi}$$

which is impossible. Indeed, we recall that

$$Q(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu},$$

and that $Q(z) \geqq 0$ on $|z| = 1$ with at least one zero there, which must be of even order. Similarly for each φ we have

$$Q^*(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu^*}{z^\nu},$$

with the same properties on $|z| = 1$. Namely $Q^*(z) \geqq 0$ on $|z| = 1$. with at least one zero there, which must be of even order. We write

$$Q^*(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{(n-\nu)\,\sigma_{n-\nu}\, e^{-i\,(1-\frac{\nu}{n-1})\varphi}}{z^\nu} + B_0^*,\ B_0^* = B_0$$

and consider the expression

$$\widetilde{Q}^*(z) = \sum_{\nu=-(n-1)}^{n-1} \frac{(n-\nu)\ \sigma_{n-\nu}}{\zeta^{1-\frac{\nu}{n-1}}\, z^\nu} + B_0^*,\ \zeta = |\zeta|\, e^{i\varphi}$$

which concides with $Q^*(z)$ on $|\zeta| = 1$. If $|z_0| = 1$ is a zero of order m of $Q(z)$, then in wiew of a fundamental theorem, in a sufficiently small neigborhood of z_0, $\widetilde{Q}^*(z)$ has m distinct roots which are analytic functions of ζ and tending to z_0 as $\zeta \to 1$ ($\varphi \to 0$). It follows that for $|\zeta| = 1$ and φ sufficiently small, $Q^*(z)$ has on $|z|=1$, zeros of order at most 1, thus contradicting the property of $Q^*(z)$ having on $|z| = 1$ at least one zero of even order.

Next, we consider neighboring cross-sections π^* as follows. By hypothesis $(\sigma_2, \sigma_3, \ldots, \sigma_{n-1})$ being an interior point of V_{n-1} there exists a function

$$f(z) = z + b_2 z^2 + \ldots + b_{n-1} z^{n-1} + b_n z^n + \ldots$$

of class S such that

$$\sigma_\nu = b_\nu, \qquad \nu = 2, \ldots, n-1,$$

and which is bounded. Hence the point

$$b = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, b_n) \tag{6}$$

is an interior point of V_n and lies in π. It follows that $|b_n| < \dot{\sigma}_n$. Let us consider the neighboring function

$$f^*(z) = e^{-i\varepsilon} f(e^{i\varepsilon} z) = z + \sigma_2^* z^2 + \ldots + \sigma^*_{n-1} z^{n-1} + b^* z^n + \ldots$$

which is also of class S for ε real, and belongs to the interior point

$$(\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1}, b_n^*)$$

where

$$\sigma_\nu^* = \sigma_\nu e^{i(\nu-1)\varepsilon}, \quad \nu = 2, \ldots, n-1; \quad b_n^* = b_n e^{i(n-1)\varepsilon}.$$

If ε is sufficiently small, then the point

$$b^* = (\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1}, b_n)$$

will be also an interior point of V_n.

Property III. Let $p = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \dot{\sigma}_n)$. Suppose that $b = (\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, b_n)$ is an interior point of V_n. Let $p^* = (\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1} \widetilde{\sigma}_n)$ be any boundary point of V_n sufficiently near p. Then each point of $b^* p^*$, save p^*, is an interior point of V_n.

Indeed, b being an interior point of V_n, if p^* is sufficiently near p and therefore ε sufficiently small, then $b^* = (\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1}, b_n)$ is interior point of V_n. The conclusion follows by applying the argument as in property II to the segment b^*p^* lying in the 2-dimensional cross-section π^*, obtained by holding $\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1}$ fixed. Namely, assuming $\mathrm{Re}\, \widetilde{\tau^0_n} < \mathrm{Re}\, \widetilde{\sigma}_n$, where $\widetilde{\tau^0_n}$ is the last coordinate in $p^* = (\sigma_2^*, \sigma_3^*, \ldots, \sigma^*_{n-1}, \widetilde{\tau^0_n})$ on $b_n\widetilde{\sigma}_n$, near p^*, we see, repeating word for word the argument at the end of the proof of property II that each point of $b^* p^*$, save p^*, is an interior point of V_n.

We also conclude that if p^* is sufficiently near p, then p^* is the only boundary point on $b^* p^*$.

It should be noticed however that property III is an immediate consequence of property II. In fact, let in general $a_n \varepsilon \Pi$. To a_n will correspond in π, the point

$$a = (\sigma_2, \sigma_3, \dots, \sigma_{n-1}, a_n)$$

If ε is real, the point

$$a^* = (\sigma_2^*, \sigma_3^*, \dots, \sigma^*_{n-1}, a_n^*)$$

where

$\sigma_\nu^* = \sigma_\nu e^{i(\nu-1)\varepsilon}$, $\nu = 2,\dots, n-1$; $a_n^* = a_n e^{i(n-1)\varepsilon}$, will lie in π^*, while a_n^* which is the projection of a^*, will lie in the projection Π^* of π^*. In fact Π^* is obtained from Π through a rotation equal to $(n-1)\varepsilon$, Thus in the neighborhood of $\sigma_n^* = \dot{\sigma}_n e^{i(n-1)\varepsilon}$ the boundary of Π^* is a continuous arc Γ^* containing σ^*_n and which is obtained from Γ through a rotation equal to $(n-1)\varepsilon$. Similarly, the boundary of π^* in the neighborhhood of

$$p^* = (\sigma_2^*, \sigma_3^*, \dots, \sigma^*_{n-1}, \sigma_n^*)$$

is a continuous arc γ^* of which Γ^* is the projection. Taking ε sufficiently small so that

$$b^* = (\sigma_2^*, \sigma_3^*, \dots, \sigma^*_{n-1}, b_n)$$

is an interior point of V_n, one obtains property III.

Property IV. In the neighborhood of $\dot{\sigma}_n$, Γ lies on both sides of the real axis, i. e., it contains points $a_n \varepsilon \Gamma$ with Im $a_n < 0$ as well as points $a_n \varepsilon \Gamma$ with Im $a_n > 0$.

Indeed, otherwise Γ will contain two arcs Γ_1, Γ_2 ending at the point $\dot{\sigma}_n$ and lying say in the upper half plane. Let Δ be a line parallel to the real axis sufficiently near to it, and which intersects Γ_1, and Γ_2 at the points a^1_n, a^2_n respectively with, say,

$$\text{Re}\, a^1_n < \text{Re}\, a^2_n, \ \text{Im}\, a^1_n = \text{Im}\, a^2_n \tag{7}$$

Applying Teichmüller's Principle to the boundary point

$$a_1 = (\sigma_2, \sigma_3, \dots, \sigma_{n-1}, a^1_n)$$

we have

$$\text{Re}\, \{(a^2_n - a^1_n)\, A^1_{n-1}\} < 0, \quad a^2_n \neq a^1_n. \tag{8}$$

In view of Lemma I, as Δ tends to the real axis, $A^1_{n-1} \to 1$. Thus it follows readily that for Δ sufficiently near to the real axis the inequality (7) is preserved and (8) is impossible . In fact, to see this, it will suffice to write (8) under the form

(9) $\operatorname{Re} a^2_n - \operatorname{Re} a^1_n < (\operatorname{Im} a^2_n - \operatorname{Im} a^1_n) \tan \arg A^1_{n-1}$.

We shall denote by Γ_1, the continuous arc with $\operatorname{Im} a_n \leqq 0$, where $\operatorname{Im} a_n = 0$ if and only if $a_n = \dot{\sigma}_n$.

Let us consider again the convex region II. We recall that Γ_1 lies, except for $\dot{\sigma}_n$, entirely in G. It will be convenient to introduce the following notations. Let $\zeta = e^{i(n-1)\varepsilon}$, where ε is assumed to be positive and sufficiently small. Denote by $\tau = \overline{\zeta}\, \dot{\sigma}_n$ the point on the arc of circumference g of G and by τ_n, the point on Γ_1 such that

$$\operatorname{Im}\tau_n = \operatorname{Im}\tau = -\dot{\sigma}_n \sin (n-1)\, \varepsilon = -\dot{\sigma}_n (n-1)\, \varepsilon + 0\,(\varepsilon^3),\ 0(\varepsilon^3)>0.$$

Let $a^0_n \;\varepsilon\; \Gamma_1$ such that $\operatorname{Re} a^0_n = \operatorname{Re} \tau$. Let δ be any direction issuing from τ and lying within the right angle determined by the vertex τ on g and by the segments $\tau\tau_n$ and $\tau\, a^0_n$ parallel to the axes of coordinates. Let a_n be the intersection of δ with Γ_1. If α, $0 < \alpha < \frac{\pi}{2}$, is the angle defined by δ and $\tau\tau_n$, then as α varies in the open interval $(0, \frac{\pi}{2})$, the point $a_n \;\varepsilon\; \Gamma_1$ sweeps the open subarc Γ_1^{ε} of Γ_1 extending from τ_n to a^0_n. Note that α is independent of ε. On rotating Γ_1 through ζ, we obtain the arc $\Gamma_1^* = \zeta\,(\Gamma_1)$ and the direction $\delta^* = \zeta(\delta)$ issuing from $\dot{\sigma}_n$ will lie on the upper half-plane for α fixed and $\varepsilon > 0$ sufficiently small. δ^* intersects Γ_1^* at the point $a_n^* = \zeta\, a_n$. We then have the following important consequence.

Property V. Let

$a^* = (\sigma_2^*, \ldots, \sigma^*_{n-1}, a_n^*)$, $a_n^* = a_n^*\ (\varepsilon, \alpha)$, $\sigma_\nu^* = \sigma_\nu\, e^{i(\nu-1)\varepsilon}$ $\nu = 2, \ldots, n-1$, $a_n^* = \zeta\ a_n$ with $\varepsilon > 0$ sufficiently small, a^* satisfying a differential equation ϑ_n^* of the form (loc. cilt. p. 36)

$$\left(\frac{z}{w}\ \frac{dw}{dz}\right)^2\ P^*(w) = Q^*(z)$$

where

$$P^* (w) = \sum_{\nu=1}^{n-1} \frac{A_\nu^*}{w^\nu}$$

$$Q^* (z) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu^*}{z^\nu}, \; B^*_{-\nu} = \overline{B}_\nu^*$$

and

(A) $$A_\nu^* = \sum_{k=\nu+1}^{n} \sigma_k^{*(\nu+1)} F_k^*$$

$$B_\nu^* = \sum_{k=1}^{n-\nu} k\sigma_k F^*_{k+\nu}, \; \nu = 1, 2, \ldots, n-1$$

(B)

$$B_0^* = \sum_{k=2}^{n} (k-1) \sigma_k^* F_k^*$$

Then for fixed α, $Q^* (z)$ which is analytic in z is continuous with respect to ε ($\varepsilon \geqq 0$) and z, and $Q^* (z) \to Q(z)$ uniformly as $\varepsilon \to 0$. Here $Q(z)$ is the right hand side of the differential equation ϑ_n corresponding to p, ($\varepsilon = 0$).

Proof. Π being convex, it follows that the arc Γ, in a neighborhood $N(\dot{\sigma}_n)$ of $\dot{\sigma}_n$, is convex, and it is well known that in $N(\dot{\sigma}_n)$, Γ is differentiable, save a countable number of points[1]. Hence if α_0, α_1 are two fixed numbers such that $0 < \alpha_0 < \alpha_1 < \frac{\pi}{2}$, it follows that for $\varepsilon > 0$ sufficiently small, and $\alpha_0 \leqq \alpha \leqq \alpha_1$, so that the points $(\varepsilon, \alpha) \in \Gamma_1$ lie in $N(\dot{\sigma}_n)$, the first partial derivatives of $\beta_n (\varepsilon, \alpha)$ and $\lambda_n (\varepsilon, \alpha)$ exist at points where Γ_1 is differentiable. Here

$$a_n = \alpha_n + i\beta_n, \; \alpha_n = \dot{\sigma}_n - \lambda_n; \; a_n^* = \alpha_n^* + i\beta_n^*, \; \alpha_n^* = \dot{\sigma}_n - \lambda_n^*$$

where

$$\beta_n, \lambda_n \to 0 \text{ as } \varepsilon \to 0.$$

On using the relations

1) G. Valiron, Théorie des Fonctions, Masson et C[ie] editeurs, 1948, pp. 79–80.

(10) $$a_n = \bar{\zeta} a_n^* = \bar{\zeta}(\dot{\sigma}_n - \lambda_n^* + i\beta_n^*)$$

and

(11) $$\beta_n^* = \lambda_n^* \tan (\alpha - (n-1)\varepsilon)$$

we find for $\alpha_0 \leqq \alpha \leqq \alpha_1$ and $\varepsilon \geqq 0$

(12) $$\lambda_n^* = (\beta_n + \dot{\sigma}_n \sin (n-1)\varepsilon) \cos (\alpha-(n-1)\varepsilon) / \sin\alpha,$$
$$\beta_n^* = (\beta_n + \dot{\sigma}_n \sin (n-1)\varepsilon) \sin (\alpha-(n-1)\varepsilon) / \sin\alpha.$$

For $\varepsilon > 0$ sufficiently small, formulas (12) define in the upper half-plane a curvilinear triangular region T with one vertex at $\dot{\sigma}_n$ and whose closure denoted by $\overline{T}$, lies, except for $\dot{\sigma}_n$, entirely in the upper half-plane. It follows that on $\overline{T}$, λ_n^* (ε, α) and β_n^* (ε, α) have first partial derivatives with respect to ε and α, save a countable number of points. Namely, formulas (12) define, by means of a system of two-dimensional cross-sections $\{\pi^*\}$, on the boundary of V_n, near p, a curvilinear triangular set of points

$$a^* = (\sigma_2^*, \ldots, \sigma^*_{n-1}, a_n^* (\varepsilon, \alpha)),$$

with one vertex at p, which we may denote by $R = R(p)$, and whose closure we indicate by $\overline{R}$, and such that $\overline{R}$ is differentiable except at a countable number of points. But in $\overline{R}$ we may write (loc. cit p. 110).

(13) $$A_\nu^* = \sum_{k=\nu+1}^{n} \sigma_k^{*(\nu+1)} F_k^*, \ \nu=1, 2, \ldots, n-1, \ \sigma_n^* = a_n^*$$

The system (13) being linear in the F_ν^* with non-vanishing determinant (loc. cit. p. 110) it can be solved for the F_ν^* in terms of $A^* = (A_1^*, A_2^*, \ldots, A^*_{n-1})$ and a^*. But, as in the proof of lemma I, the vector A^* is continuous at each point of $\overline{R}$, i. e., on $\overline{R}$. Hence F_ν^* is continuous on $\overline{R}$; and at those points where $\overline{R}$ is differentiable, we have (loc . cit. p. 111)

$$\mathrm{Re}\ \{F_2^* \, \delta\sigma_2^* + \ldots + F^*_{n-1} \, \delta\sigma^*_{n-1} + F_n^* \, \delta a_n^*\} = 0$$

Namely, the vector

$$\overline{F^*} = (\overline{F_2^*}, \ldots, \overline{F^*_{n-1}}, \overline{F_n^*})$$

is normal to V_n at the points $a^* \varepsilon \overline{R}$. Since $\overline{F^*}$ is continuous on $\overline{R}$, it follows that

(i) $\overline{R}$ is continuously differentiable at all points $a^* \varepsilon \overline{R}$ and

(ii) the vector $(0, 0, \ldots, 0, 1)$ being the value of $\overline{F^*}$ at the point p, then

$$(14) \qquad \overline{F^*} = (\overline{F_2^*}, \ldots, \overline{F^*_{n-1}}, \overline{F_n^*}) \rightarrow (0, 0, \ldots, 0, 1)$$

as $\varepsilon \rightarrow 0$, uniformly with respect to α, $\alpha_0 \leqq \alpha \leqq \alpha_1$.

Hence, $Q^*(z)$ is continuous on $\overline{R}$, and for each fixed $\alpha_0 \leqq \alpha \leqq \alpha_1$, $Q^*(z) \rightarrow Q(z)$ uniformly as $\varepsilon \rightarrow 0$.

In view of the relation $a_n = \zeta a_n^*$, where a_n^* has by (i) continuous first partial derivatives, a_n itself has continuous first partial derivatives with respect to ε, α. Hence repeating the same argument for Γ_2, namely, when $a_n \varepsilon \Gamma_2$ is the intersection of δ with Γ_2, one concludes that Γ is continuously differentiable in $N(\dot{\sigma}_n)$.

Finally, using s, the arc length of Γ_1 from $\dot{\sigma}_n$ to a_n, as the parameter to fix the position of $a = (\sigma_2, \ldots, \sigma_{n-1}, a_n)$, $a_n = a_n(s)$, by fixing the position of a_n on Γ_1, we see that along γ_1,

$$Q(z, s) = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu(s)}{z^\nu}, \qquad \overline{B_\nu}(s) = B_{-\nu}(s)$$

where $Q(z,s)$ is continuous in both variables, and $Q(z,s) \rightarrow Q(z)$ as $s \rightarrow 0$. First, we note that on Γ_1, the polynomial

$$z^{n-1} Q(z,s) = \overline{B}_{n-1}(s) z^{2n-2} + \ldots + B_0(s) z^{n-1} + \ldots + B_{n-1}(s),$$
$$B_{n-1}(s) \neq 0, \; B_0(s) > 0,$$

has the same zeros on $|z| = 1$ as $Q(z, s)$. Next, the expression on the right hand side, when considered as a polynomial R in z and the real variables x_ν, y_ν defined by

$$B_\nu = x_\nu + iy_\nu, \quad \overline{B}_\nu = x_\nu - iy_\nu, \; \nu = 1, \ldots, n-1, \; B_0 = x_0 > 0$$

is linear with respect to these $2n-1$ independent real variables and therefore irreducible within the *formal* polynomials ordered according to decreasing powers of z, with polynomials in the $2n-1$ variables x_ν, y_ν, as coefficients. Hence R and its partial derivative R_z with respect to z are relatively prime. Consequently[1], the discriminent

$$D = D\,(x_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1})$$

is not identically zero in the $2n-1$ real Euclidean space E^{2n-1}. Next, eliminating the F_k's amongst the relations (B) as applied to a, we obtain the equation

$$(B^*) \qquad B_0 = \sigma_2 B_1 + (2\sigma_3 - 2\sigma_2^2)\, B_2 + (3\sigma_4 - 7\sigma_2\sigma_3 + 4\sigma_2^3)\, B_3 + \ldots + \{(n-1)\, a_n + \varphi_{n-1}(\sigma_2, \ldots, \sigma_{n-1})\}\, B_{n-1},$$

where φ_{n-1} is a polynomial in $\sigma_2, \ldots, \sigma_{n-1}$.

If $\sigma_2, \ldots, \sigma_{n-1}$ are fixed, so is φ_{n-1} and it follows from (B*) and (B) that there is a one to one continuous correspondence between the points $a_n(s)$ and the points

$$B(s) = (B_0(s), B_1(s), \ldots, B_{n-1}(s)) \in E^{2n-1}.$$

In fact the equality $B\,(s) = B\,(s')$ implies upon substraction

$$(n-1)\;(a_n(s) - a_n(s'))\; B_{n-1} = 0.$$

Then, since $n > 1$, $B_{n-1} \neq 0$, it follows that $a_n\,(s) = a_n\,(s')$.

Consequently, as $a_n\,(s)$ describes Γ_1, $B(s)$ describes in a one to one continuous way an arc $\varkappa_1$ in E^{2n-1}.

Now, at each point of $\varkappa_1$, B_0. B_ν satisfy a linear relation of the form

$$(B^{**}) \qquad B_0 + \min_{|z|=1} \left\{ \sum_{\nu=1}^{n-1} \left(\frac{B_\nu}{z^\nu} + \overline{B}_\nu z^\nu \right) \right\} = 0,$$

or,

$$x_0 + 2 \min_{0 \leqq \theta \leqq 2\pi} \sum_{\nu=1}^{n-1} (x_\nu \cos \nu\theta + y_\nu \sin \nu\theta) = 0,$$

1) M. Bôcher, Introduction to Higher Algebra. The Macmillan Company N.Y., 1907, pp. 212—213.

where the minimum occurs at a multiple zero, say $z = z_0$, $z_0 = z_0(s)$ $|z_0| = 1$, of $Q(z,s) = 0$, with $z_0(s)$ continuous in s[1]. More generally, since each $a^* \varepsilon \overline{R}$ determines uniquely F^*, we see from (B) that the vector B^* is uniquely determined by a^* and that the correspondence $a^* \rightarrow B^*$ is continuous on $\overline{R}$. Next, the components of a^* can be calculated from (B) step by step in terms of the components of B^* and F^* with the conclusion that on $\overline{R}$, $\sigma_\nu^* = \sigma_\nu^{*\prime}$ implies $B^*_{n-\nu} = B^{*\prime}_{n-\nu}$. Hence the inverse mapping $B^* \rightarrow a^*$ so defined is one to one. We conclude that $a^* \rightarrow B^*$ *is one to one continuous on* $\overline{R}$. Hence closed sets are mapped onto closed sets. Consequently, the inverse mapping $B^* \rightarrow a^*$ is also continuous. It follows that *the vector B^* describes in E^{2n-1} a region $\overline{N}$ that is the topological image of $\overline{R}$.* At each point of $\overline{N}$, B^* satisfies a linear relation of the form (B^{**}) where the minimum occurs at a multiple zero of $Q^*(z)$, on $|z| = 1$.

But, $D = 0$ being necessary and sufficient for $Q^*(z) = 0$ to have a multiple root on $|z| = 1$, it follows that D vanishes on $\overline{N}$. Geometrically, if z is considered as a parameter then $D = 0$ is simply the envelope of the hyperplanes $R = 0$. However, $D = 0$ is real, irreducible and homogeneous of degree $4n$-6, i. e., an algebraic hypersurface. Now, $\overline{N}$ is that portion of $D = 0$ at each point of which (B^{**}) is the tangent hyperplane. Since $\overline{N}$ is algebraic, there exists on it an arc σ with one end point at $B(0)$ along which B^* can be expressed analytically with respect to some parameter and $Q^*(z) \rightarrow Q(z)$. But this contradicts the fact that $Q^*(z)$ should have multiple zeros along σ, near $B(0)$. Thus no such $\overline{N}$ and thereby $\varkappa_1$ can exist unless reduced to the point $B(0)$. Hence Γ_1 reduces to the point $\dot{\sigma}_n$, and theorem I is proved.

We illustrate this by an example corresponding to the classical case $n = 2$. In thise case $|\sigma_2| = 2$. We may assume $\sigma_2 = 2$. Then, $D = 0$, $R = 0$, $R_z = 0$ all coincide whenever B_1 is real and $z = -1$.

1) By the fundamental theorem on the continuity of the roots, $\varkappa_1$ cannot be an analytic arc.

In fact we have $B_0 = 2 B_1$. Then comparing with (B^*) we obtain complete identity, since $\sigma_2=2$. Also, $B_1=1$, $B_0=2$ as expected.

Corollary. If $n = 3$, theorem I implies that σ_2 must be a boundary point of V_2, i. e. $|\sigma_2| = 2$. Hence the extremal function corresponding to the third coefficient is the Koebe function. Accordingly, $|a_3| \leqq 3$.

Theorem II. Let $p = (\sigma_2, \ldots, \sigma_{n-1}, \dot{\sigma}_n)$, $\dot{\sigma}_n > 0$. Then p belongs to the Koebe function with $\dot{\sigma}_n = n$.

Proof. Since by Theorem I, the point $(\sigma_2, \ldots, \sigma_{n-1})$ is a boundary point of the coefficient region V_{n-1}, it satisfies a differential equation of the form

$$(15) \qquad \left(\frac{z}{w}\frac{dw}{dz}\right)^2 \sum_{\nu=1}^{n-2} \frac{A_\nu'}{w^\nu} = \sum_{\nu=-(n-2)}^{n-2} \frac{B_\nu'}{z^\nu}, \quad w=\sigma(z).$$

As in the proof of Lemma I the boundary of $\sigma(z)$ is continuous at $A' = (A_1', A_2', \ldots, A'_{n-2})$ which implies $A'_{n-2} = B'_{n-2} \neq 0$ (loc. cit. pp. 81-87). On the other hand $\sigma(z)$ satisfies the differential equation ϑ_n, i.c.,

$$(16) \qquad \left(\frac{z}{w}\frac{dw}{dz}\right)^2 \sum_{\nu=1}^{n-1} \frac{A_\nu}{w^\nu} = \sum_{\nu=-(n-1)}^{n-1} \frac{B^\nu}{z^\nu}, \quad A_{n-1} = B_{n-1} = 1.$$

Eliminating $\left(\frac{z}{w}\frac{dw}{dz}\right)$ between the two differential equations, we obtain

$$(17) \qquad w\,\frac{A_1' w^{n-3} + \ldots + A'_{n-2}}{A_1 w^{n-2} + \ldots + A_{n-1}} = z\,\frac{\overline{B}'_{n-2} z^{2n-4} + \ldots + B'_{n-2}}{\overline{B}_{n-1} z^{2n-2} + \ldots + B_{n-1}}$$

Thus w is an algebraic function of z and to each value of z there corresponds at most n—2 values of w. From either of the given differential equations it follows that $w \to 0$ as $z \to 0$. Equation (17) can be writtten as

$$(18) \qquad w\{\lambda_0 + \lambda_1 w + \ldots\} = z\{\mu_0 + \mu_1 z + \ldots\}, \lambda_0 = \mu_0 \neq 0.$$

Thus in a neighborhood of the origin each branch of $w = w(z)$ is analytic and has an expansion

(19) $$w = z + \beta_2 z^2 + \dots$$

Hence all branches of $w(z)$ coincide in a neighborhood of the origin thus implying that all branches of $w(z)$ coincide for all z, and $w(z)$ is single-valued and therefore rational

Accordingly
$$w = z\frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are polynomials without common factors and $P_1(0) = P_2(0) = 1$. Following Schaeffer-Spencer (loc. cit. pp. 156 — 158), we see that $P_1(z)$ is a constant and $P_2(z)$ is a polynomial of precise degree 2. Thus

$$w = \frac{z}{1 + \lambda z + \mu z^2}, \qquad |\mu| = 1.$$

Since the product of zeros of $1 + \lambda z + \mu z^2$ is of modulus 1 and no zero can lie in $|z| < 1$ it follows that both zeros lie on $|z| = 1$ and consequently ,

(20) $$w = \frac{z}{(1 - e^{i\alpha} z)(1 - e^{i\beta} z)}.$$

Finally $w = \sigma(z)$ being extremal, (20) reduces to Koebe function with $\dot{\sigma}_n = n$.

Exactly as in case $n = 2$, we verify, for any n, that $R = 0$, $R_z = 0$, and $D = 0$ thereof, coincide whenever B_ν is real and $z = -1$. Namely.

$$B_0 - 2B_1 + 2B_2 - \dots \pm 2B_{n-1} = 0.$$

Comparing this with (B*) we obtain complete identity, since by theorem II,

$$\sigma_2 = 2, \ \sigma_3 = 3, \ \dots, \ \dot{\sigma}_n = n.$$

Also, $B_0 = n(n-1)$, $B_\nu = (n-\nu)^2$.

We collect all these as

Corollary. A boundary point q of V_n, $n \geq 2$, is extremal and belongs to the Koebe function $f(z) = \frac{z}{(1-z)^2}$, if and only if the vector $B = (B_0, B_1, \dots, B_{n-1})$ associated with q satisfies (B**) with $B_0 > 0$, B_ν, $\nu = 1, \dots, n-1$, real and with minimum occuring at $z = -1$.

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ÖZET[1)]

2– boyutlu arakesit metodu ile gösteriliyor ki,

$$f(z) = z + a_2 z^2 + \dots, \quad |z| < 1$$

analitik ve schlicht fonksiyonları için

$$|a_n| \leqq n$$

eşitsizliği daima doğrudur. Herbir n için, $n \geqq 2$, eşitlik yalnız ve yalnız Koebe fonkiyonu için vardır.

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