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Acceleration Axes in Spatial Kinamatics I.

by

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Acceleration Axes in Spatial Kinamatics I.

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ABSTRACT

Section I is introduction. In section II we give the basic concepts for the method of Study mapping. In section III, it is shown that there are, in general, three acceleration axes in three dimensional line space. Their location and reality are derived. These axes, in general, are skew lines, and the spherical case, derived by Bottema, is the special case that three axes form a pencil of lines.

I. INTRODUCTION

The acceleration of a point in spherical motion is the sum of three orthogonal components: (a) a component normal to the sphere; (b) a component tangent to the path (or tangential acceleration); and (c) a component normal to the path but lying in the plane tangent to the sphere (geodesic normal acceleration). The locus of points having zero tangential acceleration is the spherical equivalent of the Bresse circle of planar kinematics. Following Garnier's notation we denote this spherical curve as C_2. The locus of points having zero geodesic normal acceleration is the spherical equivalent of the inflection circle, denoted as C_3. The intersections of the curve C_2 and C_3 have neither a tangential acceleration nor a geodesic normal acceleration: their acceleration is purely normal to the sphere and they are the acceleration centers S_i discussed in the paper of Bottema [1].

The oriented lines in R^3 are in one-to-one correspondence with the points of the *dual unit sphere* D^3 (E. Study). Using this correspondence, in this paper we extend the spherical properties

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discovered by Bottema to three dimensional line space R^3. In order to do this we consider that the sphere is a unit dual sphere, hence everything belonging to the sphere is dual: dual point, dual angle, dual vector, dual number [2]. A dual point of D^3 corresponds to a line in R^3; two different points of D^3 represent two skew lines in R^3. A differentiable curve on D^3 represents a ruled surface in R^3. In section II we give the basic concepts of this method.

In section III, it is shown that there are, in general, three acceleration axes in R^3. Their location and reality are derived. These axes, in general, are skew lines and the spherical case is the special case that three axes form a pencil of lines.

II. BASIC CONCEPTS

a) Dual Numbers, Dual Vectors and Dual Angles:

Definition (2-1): If a and a* are real numbers and $\varepsilon^2=0$, the combination

$$A = a + \varepsilon a^* \tag{2-1}$$

is called a *dual number*.

Hence ε is the dual unit. Dual numbers are considered as polynomials in ε, subject to the defining relation $\varepsilon^2 = 0$. W. K. Clifford defined the dual numbers and showed that they form an algebra, not a field. The *pure dual* numbers are εa^*. According to the definition pure dual numbers εa^* are zero divisors, $(\varepsilon a^*)(\varepsilon b^*) = 0$. No number εa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as the laws of the algebra of complex numbers $(a+ib, i^2 = -1)$. This means dual numbers form a ring over the real number field. For example two dual numbers A and $B=b+\varepsilon b^*$ are added componentwise.

$$A + B = (a + b) + \varepsilon(a^* + b^*) \tag{2-2}$$

and they are multiplied by

$$A B = a b + \varepsilon(a^* b + a b^*). \tag{2-3}$$

For the equality of A and B we have

$$A = B \rightleftarrows a = b \quad \text{and} \quad a^* = b^*. \tag{2-4}$$

An oriented line in R^3 may be given by two points on it, $\vec{x}$ and $\vec{y}$. If p is any nonzero constant, the parametric equation of the line is

$$\vec{y} = \vec{x} + p\,\vec{a} \tag{2-5}$$

and the moment of the vector $\vec{a}$ with respect to the origin 0 is

$$\vec{a}^* = \vec{x} \times \vec{a} = \vec{y} \times \vec{a}\,. \tag{2-6}$$

This means that the direction vector $\vec{a}$ of the line and its moment vector $\vec{a}^*$ are independent of the choice of the points of the line. The two vectors $\vec{a}$ and $\vec{a}^*$ are not independent of one another; they satisfy the following equations:

$$\vec{a}^2 = 1, \qquad \vec{a}.\ \vec{a}^* = 0. \tag{2-7}$$

The six components a_i, a^*_i (i=1, 2, 3) of $\vec{a}$ and $\vec{a}^*$ are *Plückerian homogeneous line cooordinates.* Hence the two vectors $\vec{a}$, $\vec{a}^*$ determine the oriented line. A point $\vec{z}$ is on the line of vectors $\vec{a}$, $\vec{a}^*$ if and only if

$$\vec{z} \times \vec{a} = \vec{a}^*. \tag{2-8}$$

The set of oriented lines in R^3 is in one-to-one correspondence with pairs of vectors in R^3 subject to the conditions (2-5), and so we may expect to represent it as a certain four-dimensional set in R^6 of sixtuples of real numbers; we may take the space D^3 of triples of dual numbers with coordinates

$$X_1 = x_1 + \varepsilon x^*_1,\ X_2 = x_2 + \varepsilon x_2^*,\ X_3 = x_3 + \varepsilon x_3^*. \tag{2-9}$$

Each line in R^3 is represented by the *dual vector* in D^3

$$\vec{A} = \vec{a} + \varepsilon\vec{a}^* \quad (2\text{-}10)$$

$\vec{A}$ is a *dual unit vector*

$$\vec{A} \cdot \vec{A} = \vec{a} \cdot \vec{a} + 2\varepsilon\vec{a} \cdot \vec{a}^* = 1 \quad (2\text{-}11)$$

if we carry over the formal definition of the products of vectors to dual space.

Theorem (2-1) (E. Study). The oriented lines in R^3 are in one-to-one correspondence with the points of the *dual unit sphere*

$\vec{A} \cdot \vec{A} = 1$ in D^3 [2].

The scalar product of two dual vectors $\vec{A} = \vec{a} + \varepsilon\vec{a}^*$ and $\vec{B} = \vec{b} + \varepsilon\vec{b}^*$ is

$$\vec{A} \cdot \vec{B} = \vec{a} \cdot \vec{b} + \varepsilon (\vec{a}^* \cdot \vec{b} + \vec{a} \cdot \vec{b}^*). \quad (2\text{-}12)$$

Both $\vec{a}$ and $\vec{b}$ are unit vectos; $\vec{a} \cdot \vec{b}$ is the cosine of the angle φ of the two lines which correspond to $\vec{A}$, $\vec{B}$. the skew lines in space have a unique common perpendicular (Fig. 2-1).

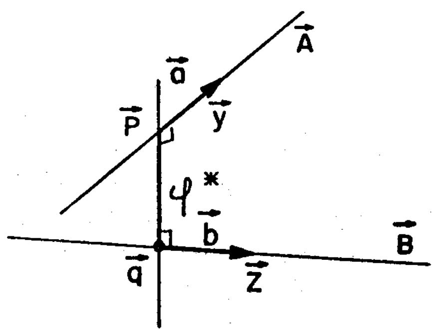

FIG. 2-1

The minimal distance φ^* between the two lines is defined as the shortest distance between the lines.

Theorem (2-2). $\vec{A}.\ \vec{B}$ is the cosine of the *dual angle* $\Phi = \varphi + \varepsilon\varphi^*$ of the two lines. $(\vec{A}.\ \vec{B} = \cos\Phi)$.

Proof: According to (2-12) we may express this theorem as follows:

$$\vec{A}.\ \vec{B} = \vec{a}.\ \vec{b} + \varepsilon(\vec{a}.\ \vec{b}^* + \vec{a}^*.\vec{b}) = \cos\Phi. \quad (2\text{-}13)$$

On the other hand, since the Taylor polynomial of an analytic dual variable function has just two terms we may write

$$f(t + \varepsilon h) = f(t) + \varepsilon f'(h)$$

$$\left.\begin{aligned} \sin\Phi &= \sin(\varphi+\varepsilon\varphi^*) = \sin\varphi + \varepsilon\varphi^* \cos\varphi \\ \cos\Phi &= \cos(\varphi+\varepsilon\varphi^*) = \cos\varphi - \varepsilon\varphi^* \sin\varphi \end{aligned}\right\} . \quad (2\text{-}14)$$

Then (2-4) and (2-13) give us

$$\left.\begin{aligned} \vec{a}\ .\ \vec{b} &= \cos\varphi\ , \\ \vec{a}.\vec{b}^* + \vec{a}^*.\vec{b} &= -\varphi^* \sin\varphi . \end{aligned}\right\} \quad (2\text{-}15)$$

Since φ is the angle of the two lines the first equality of (2-15) is obvious. In order to show the second equality of (2-15), we choose points $\vec{y} = \vec{p} + \vec{a}$ on $\vec{A}$, $\vec{z} = \vec{q} + \vec{b}$ on $\vec{B}$. Then

$$\vec{a}\ .\ \vec{b} + \vec{a}^*.\vec{b} = (\vec{a}, \vec{q}, \vec{z}) + (\vec{p}, \vec{y}, \vec{b}) = - (\vec{a}, \vec{b}, \vec{q}) +$$

$$+ (\vec{a}, \vec{b}, \vec{p}) = (\vec{a}, \vec{b}, \vec{p} - \vec{q}) = - \varphi^* \sin\varphi .$$

Here the last equation follows from the interpretation of the determinant as the (oriented) volume of the parallelepiped spanned by its row vectors.

Hence (2-13) and (2-14) give

$$\vec{A}.\ \vec{B} = \cos\Phi = \cos\varphi - \varepsilon\varphi^{*}\sin\varphi, \tag{2-16}$$

and the following special cases of (2-16) are very important in this paper:

I. If

$$\vec{A}.\ \vec{B} = 0 \tag{2-17}$$

then $\varphi = \frac{\pi}{2}$ and $\varphi^{*} = 0$; this menas that two lines $\vec{A}$ and $\vec{B}$ meet at a right angle.

II. If

$$\vec{A}.\ \vec{B} = \text{pure dual}, \tag{2-18}$$

then $\varphi = \frac{\pi}{2}$ and $\varphi^{*} \neq 0$; the lines $\vec{A}$ and $\vec{B}$ are orthogonal skew lines.

III. If

$$\vec{A}.\ \vec{B} = \text{pure real}, \tag{2-19}$$

then $\varphi \neq \frac{\pi}{2}$ and $\varphi^{*} = 0$; the lines $\vec{A}$ and $\vec{B}$ intersect.

IV. If

$$\vec{A}.\ \vec{B} = +1 \tag{2-20}$$

then $\varphi = 0$ and $\varphi^{*} = 0$; the lines $\vec{A}$ and $\vec{B}$ are coincident (their senses are the same or opposite).

b) Spatial Motions:

Since an euclidean motion in R^3 leaves unchanged the angle and the distance between two lines it will leave also unchanged the dual angle between two lines. Therefore the corresponding transformation in D^3 will leave the scalar product

$$\vec{A}.\ \vec{B} = AB^{T}$$

invariant. It is the action of an orthogonal matrix with dual coefficients. When the center of the dual unit sphere must remain fixed, the transformation group in D^3, which is the image of the euclidean motions in R^3, does not contain any translations.

Theorem 2-3. The euclidean motions in R^3 are represented in D^3 by the dual orthogonal matrices $x = (X_{ij})$, $xx^T = I$, X_{ij} dudual numbers.

The lie algebra $L(O_{D^3})$ of the group of 3x3 orthogonal dual matrices is the algebra of skew-symmetric 3x3 dual matrices. This is seen by differentiation of $xx^T = I$. Therefore we can easily extend all known formulas about real spherical motions [3] to the dual spherical motions. But it is necessary to pay attention to the zero divisors.

The two coordinate systems $\{0; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{0'; \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ are right-handed orthogonal coordinate systems which represent the *moving space H* and the *fixed space H'* respectively. Let us express the displacements (H/H') of H with respect to H' in a third orthonormal right-handed system (relative system) $\{N; \vec{r}_1, \vec{r}_2, \vec{r}_3\}$. Then the corresponding dual orthonormal coordinate axes are

$$\vec{E}_i = \vec{e}_i + \varepsilon \vec{e}^*_i,\ \vec{E}'_i = \vec{e}'_i + \varepsilon \vec{e}^{*\prime};\ \vec{R} = \vec{r}_i + \varepsilon \vec{r}^*_i, (i=1,2,3), \quad (2\text{-}21)$$

where

$$\vec{e}^*_i = \vec{OM} \times \vec{e}_i;\quad \vec{e}^*_i{}' = \vec{O'M} \times \vec{e}'_i; \vec{r}^*_i = \vec{NM} \times \vec{r}_i; (i=1,2,3) \quad (2\text{-}22)$$

and M is a fixed origin point in the space. Then the corresponding dual pfaffian forms are

$$\Omega_i = \omega_i + \varepsilon\omega^*_i,\quad \Omega'_i = \omega'_i + \varepsilon\omega^*_i{}', \ ((i=1,2,3). \quad (2\text{-}23)$$

Hence we can write the following formulas for the dual spherical motions which are equivalent to the real spherical motions [3]: a) The displacements with respect to H are

$$dR = \Omega R \quad (2\text{-}24)$$

where

$$\Omega = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix} \quad \text{and } R = \begin{bmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vec{R}_3 \end{bmatrix}. \qquad (2\text{-}25)$$

b) the displacements with respect to H′ are

$$d'R = \Omega' R \qquad (2\text{-}26$$

where

$$\Omega' = \begin{bmatrix} 0 & \Omega'_3 & -\Omega'_2 \\ -\Omega'_3 & 0 & \Omega'_1 \\ \Omega'_2 & -\Omega'_1 & 0 \end{bmatrix}. \qquad (2\text{-}27)$$

The real and dual parts of (2-24) or (2-26) correspond to the *pure rotation* and the *pure translation* of the motion H/H′, respectively. This separation is based on the following property of the theory of groups:

Theorem 2-4. The 6- parameter group of motions is the commutative product of the 3- parameter group of rotations and the 3- parameter group of translations.

III. ACCELERATION AXES

a) *Velocity in Spatial Motions*:

Consider a point X of unit dual sphere such that its coordinates with respect to the relative system are

$X_i = x_i + \varepsilon x_i^*$ (i = 1, 2, 3). Then

$$\sum_{i=1}^{3} X_i^2 = 1 \qquad (3\text{-}1)$$

and

$$\vec{X} = X^T R \qquad (3\text{-}2)$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}. \qquad (3\text{-}3)$$

A line $\vec{X}$ in space corresponds to this dual point X. The displacements of $\vec{X}$ with respect to H and H′ are

$$\left.\begin{aligned} d\vec{X} &= dX^T R + X^T dR \\ d'\vec{X} &= dX^T R + X^T d'R \end{aligned}\right\}$$

and

respectively or, from (2-24) and (2-26),

$$d\vec{X} = (dX^T + X^T\Omega)\ R \tag{3-4}$$

and

$$d'\vec{X} = (dX^T + X^T\Omega')\ R. \tag{3-5}$$

Therefore if $\vec{X}$ is fixed in H or H′ then $d\vec{X}=0$ or $d'\vec{X}=0$ and we may write

$$dX^T = X^T\ \Omega^T \tag{3-6}$$

or

$$dX^T = X^T\Omega'^T. \tag{3-7}$$

Now, suppose that $\vec{X}$ is fixed in H and let us calculate its velocity $d_f\vec{X}$ with respect to H′. Then we substitute (3-6) in (3-5) and obtain

$$d_f\vec{X} = X^T\ (\Omega'\text{-}\Omega)\ R. \tag{3-8}$$

If we define a new dual vector whose components in the relative system are

$$\Psi_i = \Omega'_i - \Omega_i, \tag{3-9}$$

then (3-8) reduces to

$$d_f\vec{X} = \vec{\Psi} \times \vec{X} \tag{3-10}$$

where

$$\vec{\Psi} = \vec{\psi} + \varepsilon\,\vec{\psi}^* \tag{3-11}$$

is the dual rotation Pfaffian vector. The real part $\vec{\psi}$ and the dual part $\vec{\psi}^*$ of $\vec{\Psi}$ correspond to the rotation motions and the translation motions of the spatial motions [4]. In order to leave out the pure translation motions we will suppose that

$$\vec{\psi} \neq 0 \tag{3-12}$$

b) Acceleration in Spatial Motions:

From (3-10) it follows that the acceleration of $\vec{X}$ is given by

$$\vec{J} = d_f^2\vec{X} = \vec{\Psi} \times (\vec{\Psi} \times \vec{X}) + \dot{\vec{\Psi}} \times \vec{X} = -\Psi^2 \vec{X} + (\vec{\Psi} \cdot \vec{X})\vec{\Psi} + \dot{\vec{\Psi}} \times \vec{X}, \tag{3-13}$$

where $\dot{\vec{\Psi}} = d\vec{\Psi}$ is the instantaneous *dual angular acceleration vector*. Equations (3-10) and (3-13) can be written in the matrix form:

$$d_f X = MX \tag{3-10)'$$

$$J = (M^2 + \dot{M})X \tag{3-13)'$$

where M, $\dot{M}$, M^2 are

$$M = \begin{bmatrix} 0 & -\Psi_3 & \Psi_2 \\ \Psi_3 & 0 & -\Psi_1 \\ -\Psi_2 & \Psi_1 & 0 \end{bmatrix}; \quad \dot{M} = dM = \begin{bmatrix} 0 & -\dot{\Psi}_3 & \dot{\Psi}_2 \\ \dot{\Psi}_3 & 0 & -\dot{\Psi}_1 \\ -\dot{\Psi}_2 & \dot{\Psi}_1 & 0 \end{bmatrix}; \tag{3-14}$$

$$M^2 = \begin{bmatrix} -\Psi^2 + \Psi_1^2 & \Psi_1\Psi_2 & \Psi_1\Psi_3 \\ \Psi_1\Psi_2 & -\Psi^2 + \Psi_2^2 & \Psi_2\Psi_3 \\ \Psi_1\Psi_3 & \Psi_2\Psi_3 & -\Psi^2 + \Psi_3^2 \end{bmatrix}; \quad M^3 = -\Psi^2 M \tag{3-15}$$

and for the higher order accelerations, the power of the matrix M has the following properties:

$$\left.\begin{aligned} M^{2n+2} &= (-1)^n \Psi^{2n} M \\ M^{2n+2} &= (-1)^n \Psi^{2n} M^2. \end{aligned}\right\} \tag{3-16}$$

Equations (3-16) are also valid for $\dot{M}$ if Ψ is replaced by $\dot{\Psi}$.

In the equations (3-13)′ we see that the components of the acceleration J of $\vec{X}$ are homogeneus linear functions the coordinates X_i (i=1,2,3) of $\vec{X}$. If we calculate the determinant D of the coefficients of (3-13)′, we obtain

$$D = - \Psi^2 \ \dot{\Psi}^2 \ \sin^2 \nabla \qquad (3\text{-}17)$$

where

$$\nabla = \alpha + \varepsilon\alpha^* \qquad (3\text{-}18)$$

is the dual angle between the vectors $\vec{\Psi}$ and $\dot{\vec{\Psi}}$. If both vectors $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ correspond to the same line of space, this line has no acceleration, in this special case D=0. After the discussion of the general case D $\neq$ 0 we shall return to this special case.

In order to extend the notion of the acceleration pole of spherical kinematics to the spatial motions we accept that the unit real sphere is a unit dual sphere and the same definition is on this dual sphere:

Definition 3-1. If a unit dual vector $\vec{X}$ of the unit dual sphere and its dual acceleration vector $\vec{J}$ are linearly dependent, the ponit $\vec{X}$ is the dual acceleration pole and the line $\vec{X}$ is the acceleration axis of the motion.

If we denote the acceleration axis $\vec{X}$ by $\vec{V}$, according to definition 3-1, from (3-13) the vector $\vec{V}$ satisfies the following equation:

$$(\vec{\Psi} . \vec{V}) \vec{\Psi} + \dot{\vec{\Psi}} x \vec{V} = \Lambda \Psi^2 \vec{V} \qquad (3\text{-}19)$$

where

$$\Lambda = \lambda + \varepsilon\lambda^* \qquad (3\text{-}20)$$

is a dual scalar. (3-19) is equivalent to three homogeneous linear equations of the coordinates X_1, X_2, X_3 of $\vec{V}$; so for the non-zero solutions, the determinant of the coefficients must be zero. Hence we have

$$\Lambda^3 - \Lambda^2 + K\Lambda - K\cos^2\nabla = 0 \qquad (3\text{-}21)$$

where

$$K = k + \varepsilon k^* = \frac{\dot{\Psi}^2}{\Psi^4} = \frac{\dot{\psi}^2}{\psi^4} - \varepsilon \frac{2\dot{\psi}(2\psi^*\dot{\psi} - \psi\dot{\psi}^*)}{\psi^5}. \qquad (3\text{-}22)$$

Since we have

$$\left.\begin{array}{l} \Lambda = \lambda + \varepsilon\lambda^*, \ \Lambda^2 = \lambda^2 + \varepsilon 2\lambda\lambda^*, \ \Lambda^3 = \lambda^3 + \varepsilon 3\lambda^2\lambda^* \\ \Psi = \psi + \varepsilon\psi^*, \Psi^2 = \psi^2 + \varepsilon 2\psi\,\psi^*, \Psi^4 = \psi^4 + \varepsilon 4\psi^3\,\psi^*, \\ \dot{\Psi} = \dot{\psi} + \varepsilon\dot{\psi}^*, \ \dot{\Psi}^2 = \dot{\psi}^2 + \varepsilon 2\dot{\psi}\,\dot{\psi}^* \\ \cos\nabla = \cos\alpha - \varepsilon\alpha^*\sin\alpha, \ \cos^2\nabla = \cos^2\alpha - \varepsilon\alpha^*\sin 2\alpha \end{array}\right\} \qquad (3\text{-}23)$$

(3-21) reduces to two equations:

$$\left.\begin{array}{l} \lambda^3 - \lambda^2 + k\lambda - k\cos^2\alpha = 0 \\ \lambda^* = \dfrac{k^*\cos^2\alpha - k\alpha^*\sin 2\alpha - k^*\lambda}{3\lambda^2 - 2\lambda + k} \end{array}\right\} \qquad (3\text{-}24)$$

(3-21) or (3-24) generally give three roots Λ_1, Λ_2, Λ_3 so there are in general three lines l_1, l_2, l_3 which we shall call *the instantaneous acceleration axes*. These axes represent the generalization of the acceleration pole of plane kinematics. These three axes are in general skew lines in space. The special case of $k^* = 0$ is very important; for this, according to (3-22) there are three possible cases: a) $\dot{\psi} = 0$, b) $2\psi^*\dot{\psi} - \psi\dot{\psi}^* = 0$, and c) $\psi^* = 0$. a) $\dot{\psi} = 0$ implies ψ = constant and $k = 0$. Therefore (3-21) has the roots $\Lambda_1 = \Lambda_2 = 0$ and $\Lambda_3 = 1$. According to (3-19) the corresponding acceleration axes are

$$l_1 = l_2 = (\vec{\Psi}.\vec{V}_{1,2}) \quad \vec{\Psi} + \vec{\dot{\Psi}} \times \vec{V}_{1,2} = 0$$

and

$$l_3 = (\vec{\Psi}.\vec{V}_3) \quad \vec{\Psi} + \vec{\Psi} \times \vec{V}_3 - \Psi^2\vec{V}_3 = 0.$$

b) $2\psi^*\psi - \psi\psi^* = 0$ gives us, by integration, $\psi^* = c_1\psi$. Then

the pitch of the instantaneous helicoidal motion is $\frac{\psi^*}{\psi} = c_1 \psi$.
Where c_1 is a real constant. The orbit of a point is a circular helix during the instantaneous motion.

c) If $\psi^* = 0$ then $\dot{\psi}^* = 0$ and again $k^* = 0$. In this case there is a fixed point on the instantaneous axis $\vec{\Psi}$. Since ψ^* is the translation part of the spatial motion, this special case is the spherical motion. In this case three axes l_1, l_2, l_3 form a pencil of lines whose vertex is the center of the sphere (the fixed point of $\vec{\Psi}$).

The spherical case has been presented by Bottema [1].

b) Reality of the Acceleration Axes:

In view of (3-24) the three λ_i (and also three λ^*_i) are either all real or two of them are imaginary. Therefore the three axes l_i either all real or at least one of them is real. In order to discuss the roots of equation (3-21), we define a new unknown L:

$$\Lambda = L + \frac{1}{3} . \qquad (3\text{-}25)$$

Then (3-21) reduces to

$$L^3 + BL + C = 0 \qquad (3\text{-}26)$$

where

$$L = \mu + \varepsilon\mu^*,\quad B = b + \varepsilon b^* = K - \frac{1}{3},\quad C = c + \varepsilon c^* =$$

$$\left(\frac{1}{3} - \cos^2\nabla\right) K - \frac{2}{27},$$

$$b = k - \frac{1}{3},\quad c = \left(\frac{1}{3} \cos^2\alpha\right) k - \frac{2}{27},\quad b^* = k^*,$$

$$c^* = k^* \left(\frac{1}{3} \cos^2\alpha\right) + k\alpha^* \sin 2\alpha . \qquad (3\text{-}27)$$

The real and dual parts of (3-26) give

$$\left.\begin{aligned} &\mu^3 + b\mu + c = 0 \\ &\mu^* = -\frac{b^*\mu + c^*}{3\mu^2 + b} . \end{aligned}\right\} \qquad (3\text{-}28)$$

The roots μ of the cubic equation of (3-28) are real (and the values of μ^* also are real) if and only if the discriminant

$$4b^3 + 27\, c^2 \leq 0$$

or from (3-27)

$$k\{4k^2 + (27\cos^2\alpha - 18\cos^2\alpha\text{-}1)\, k + 4\cos^2\alpha\} \leq 0$$

or since $k > 0$

$$f \equiv 4k^2 + 27\cos^4\alpha - 18\cos^2\alpha\text{-}1)\, k + 4\cos^2\alpha \leq 0 \ . \qquad (3\text{-}29)$$

The equation $f=0$ represents a cubic curve (C) on the $(k, \cos^2\alpha)$–plane.

According to Bottema [1] this curve has the following properties:

(a) The graph of (C) is as shown in Figure (3-1);

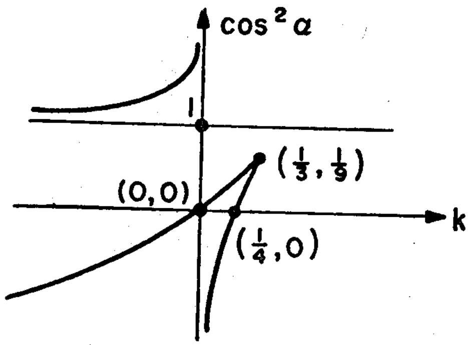

FIG. 3.1

(b) (C) has a cusp point for $k = \frac{1}{3}$, $\cos^2\alpha = \dfrac{1}{9}$, and the equation of the cuspidal tangent is

$$6k - 9 \cos^2\alpha - 1 = 0 ;$$

(c) The points of intersection of (C) and the k – axis are (0,0) and $(\frac{1}{4}, 0)$;

(d) The asymptotes of (C) are the lines $k = 0$ and $\cos^2\alpha = 1$.

The points on (C) correspond to the positions which have two parallel acceleration axes. The cusp point corresponds to the position which has three parallel acceleration axes. The other points of the strip $k > 0, 0 \leq \cos^2\alpha < 1$ correspond to the positions which have only one real acceleration axis. Hence we may express the following theorem:

Theorem 3-1. In the spatial one-parameter motion, the three acceleration axes are real if and only if (3-29) holds.

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