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Generalizations Of The Non-Desarguesian Moulton's Plane

by

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Generalizations Of The Non-Desarguesian Moulton's Plane

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ABSTRACT

This paper contains three parts. The first part is the introduction. In the second part, we construct a class of non-desarguesian planes which contains Moulton's plane as a particular case. In the third part, we show that the constructed planar geometries can be regarded as affine geometries.

I. INTRODUCTION

F. R. Multon [1] constructed a new plane by deformong the lines of the euclidean projective plane. The points of this plane are points of the euclidean projective plane. The lines of this plane are defined by

$$(1) \qquad y = f(y,m)m(x - a) ,$$

where $m,a \in R$, and $f(y, m)$ is defined as follows:

$$(2) \begin{cases} \text{If } m \in R^- + \{0\}, \text{ then } f(y,m) = 1; \\ \text{If } m \in R^+ \text{ and } y \in R^-, \text{ then } f(y,m) = 1; \\ \text{If } m \in R^+ \text{ and } y \in R^+ + \{0\}, \text{ then } f(y,m) = c,\ c \neq 1,\ c \in R^+. \end{cases}$$

Here R, R^+ and R^- are the sets of all real numbers, positive real numbers and negative real numbers, respectively. In addition, the equation of the line is given by

$$y = b \qquad (b \in R),$$

when $m = 0$ and $a \to \infty$, and by

$$x = a$$

when $m \to \infty$. That is, in this plane geometry, the lines of non-positive slope are euclidean lines, and the lines of positive slope are euclidean broken lines, broken at the x-axis so that the slope in the upper half-plane is a positive constant c (not unity) times the slope in the lower half-plane.

Later, K. Levenberg [2] generalized Moulton's non-desarguesian plane geometry by consideration of the broken line plane geometries defined by an arbitrary functional relationship between the upper and lower half-plane angles, θ and $\varnothing$, respectively. Here the function $\varnothing = f(\theta)$ is strictly monotonic increasing continuous single-valued function with a single-valued invers, in the closed interval $[0, \pi]$. All the plane geometries obtained , in this way, are constructed by the broken lines*.

In this paper, we give generalizations of Moulton's plane. To do this, in section II, we first break the parts of the straight lines in the upper half-plane by a certain ratio c and then replace their equations by particularly chosen polynomials of the odd degree. Thus, we construct the real planes such that a part of infinite lenght of their lines are deformed to the non-linear curves**. Moreover, we consider the euclidean projective plane as a particular case. In the section III, we show that the constructed planar geometries can be regarded as affine geometries at the same time.

I should like to express my indeptednes to Professor Dr. Masatoshi Ikeda, for this encouragement and advise in the preparation of this paper.

II. GENERALIZATIONS OF MOULTON'S PLANE

The points of a generalized Moulton's plane are the points of the euclidean projective plane. The lines of such a plane are given by

$$y = f(y,m)\, m\, (x-a)^{2n+1}, \qquad n \in N, \tag{3}$$

where f(y,m) is described as follows:

* There are some other non-desarguesian planes besides these, see [3].

** By non-lineer curves, we mean planar curves which are not particularly, straight or broken lines.

$$
(4)\quad \begin{cases} \text{If } m \in R^- + \{0\},\ \text{then } f(y,m) = 1 \text{ and } n = 0; \\ \text{If } m \in R^+ \text{ and } y \in R^-,\ \text{then } f(y,m) = 1 \text{ and } n = 0; \\ \text{If } m \in R^+ \text{ and } y \in R^+ + \{0\},\ \text{then } f(y,m) = c,\ c \neq 1,\ c \in R^+, \\ \qquad\qquad c = \text{constant}; \\ \text{If } m \to \infty \text{ then } f(y,m)\, m = d,\ d \in R - \{0\},\ n = 0 \text{ and } y = 0. \end{cases}
$$

Moreover, when $m = 0$ and $a \to \infty$, the equation of the line is given by

$$y = b \qquad (b = y - \text{intercept}).$$

Hereafter , in order to distinguish the lines of this plane geometry from the ones in the euclidean plane we use the term "stragiht line" instead of the term "line" in the euclidean sense.

Now, let us show that these generalized planes satisfy the axioms of alignment of a general projective geometry of a plane.

Theorem 1. There is only one line joining any two points of the plane.

Proof. It suffices to prove the theorem for the following two cases:

(i). The two points are in the upper half-plane;

(ii). While one point is in the upper half-plane the other one is in the lower half-plane.

Case i) Let the points $N_1 = (x_1, y_1)$ and $N_2 = (x_2, y_2)$ be given, where $x_1, y_1 \in R^+$, $x_2, y_2 \in R^+ + \{0\}$ and $x_1 > x_2$, $y_1 > y_2$. The value of the parameters m and a which are sufficient to define the line (3) can be calculated uniqely as follows

$$a = x_1 - \frac{x_1 - x_2}{1-(y_2/y_1)^{1/2n+1}} \quad \text{and} \quad m = \frac{y_1\,[1-(y_2/y_1)^{1/2n+1}]^{2n+1}}{c(x_1-x_2)^{2n+1}} \ ;$$

because of these points satisfy the equation

$$y = c\, m\, (x - a)^{2n+1} \quad , \qquad n \in N.$$

Similar proofs may be given for the following cases in each of which choosing the suitable coordinates x_i and y_i, $(i = 1, 2)$:

(a) Both of the N_1 and N_2 are the points of the second quadrant;

(b) One point is in the first quadrant and the other is in the second.

Case ii) Let the points $N_1 = (x_1, y_1)$ and $N = (x_2, -y_2)$ be in first and fourth quadrants, respectively; where x_1, x_2, $y_2 \in R^+$ and $y_1 \in R^+ + \{0\}$.

In accordance with the definition suppose that $x_1 > x_2$. Since these points are to satisfy the equation of the line on which they lie, the equations

$$(5) \qquad y_1 = c\, m\, (x_1 - a)^{2n+1}$$

and

$$(6) \qquad -y_2 = m\,(x_2 - a)$$

are valid. From Eq. (5) and Eq. (6) we have

$$(7) \qquad cy_2\,(a - x_1)^{2n+1} + y_1\,(a - x_2) = 0.$$

Now, if we denote $(x_1 - x_2)$ by k, $k \in R^+$, the Eq.(7) yields

$$(8) \qquad cy_2\,(a-x_1)^{2n+1} + y_1\,(a - x_1) + k\,y_1 = 0.$$

If Y denotes $(a - x_1)$, that is,

$$(9) \qquad Y = (a - x_1)$$

(8) reduces to

$$(10) \qquad Y^{2n+1} + \lambda Y + \mu = 0,$$

where $\lambda = y_1 / cy_2$, $\mu = ky_1/cy_2$, $\lambda, \mu \in R^+$. It is obvious that Eq. (10) can not have any positive real roots. For in this case the left side of the equation is definitely positive. Hence, all the real roots have to be negative. Now, if we define a new variable Z by

$$(11) \qquad Y = -Z$$

then Eq. (10) reduces to

$$(12) \qquad Z^{2n+1} + \lambda Z - \mu = 0.$$

All the real roots of Eq.(12) are positive. "Descartes' rule of signs" and some well known theorems of algebra imply the existence of one real root. Thus the parameter a, in Eq.(5), is uniqely determined by going back from Eq.(12). Then the parameter m, in Eq.(5), can be easily determined from Eq.(6).

Choosing the suitable coordinates, similar proofs can be given in the following cases:

(a) While one of the points N_1 and N_2 is in the third quadrant, the other one is in the first;

(b) While one of the points N_1 and N_2 is in the third quadrant, the other one is in the second.

Theorem 2. If the straight lines

$$L_1 \ldots y = m_1 (x-a_1), \qquad m_1 \in R$$

$$L_2 \ldots y = m_2 (x-a_2), \qquad m_2 \in R \;, \; a_1 \neq a_2$$

intersect at a point of the upper half-plane, then the half lines corresponding to L_1 and L_2, respectively,

$$C_1 \ldots y = c\, m_1 (x-a_1)^{2n+1}$$

$$C_2 \ldots y = c\, m_2 (x-a_2)^{2n+1}$$

intersect at only one point in the upper half-plane. If the straight lines L_1, L_2 intersect in the lower half-plane or if they are parallel (i. e. not intersecting at a finite point), then the intersection set of the lines C_1, C_2 is empty.

Proof. Let the straight lines L_1, L_2 intersect at the first quadrant of the upper half-plane. For this purpose, we choose $m_2 > m_1$ and $a_2 > a_1$. Then the system of equations C_1, C_2 gives the following intersection point

$$(13) \quad \left[\frac{a_2(m_2)^{1/2n+1} - a_1(m_1)^{1/2n+1}}{(m_2)^{1/2n+1} - (m_1)^{1/2n+1}},\; cm_1m_2 \left(\frac{a_2 - a_1}{(m_2)^{1/2n+1} - (m_1)^{1/2n+1}}\right)^{2n+1}\right].$$

This is a real point in the first quadrant if $a_2 \in R^+ + \{0\}$, and a real point in the first or in the second quadrant if $a_2 \in R^-$.

To prove the second part of the theorem, let the straight lines L_1, L_2 intersect at the lower half-plane. For this purpose, we can choose $m_2 > m_1$ and $a_1 > a_2$. Then the point (13) lies in the lower half-plane, since $a_2 - a_1 \in R^-$. This means that the lines C_1 and C_2 do not have a common point in the upper half-plane.

For the last part of the theorem, suppose that the line L_1 does not intersect L_2 at a finite point. In this case, we can take

$m_1 = m_2 = m$ and $a_1 \neq a_2$. Then it is easily seen that C_1 and C_2 can not have a finite common point.

Theorem 3. If the straight line $y = -m_1(x - b)$, $m_1 \in R^+$, or a straight line parallel to x or y axis, intersects the straight line $y = cm(x - a)$ in the upper half plane, then it also intersects the half line

$$y = cm(x - a)^{2n+1}, \quad y > 0$$

at a point of the upper half - plane.

Proof. The theorem is obvious for the lines parallel to the coordinate axes. Let us consider the straight line

$$y = -m_1(x - b), \quad m_1 \in R^+ \tag{14}$$

with negative slope, and the line

$$y = c\,m\,(x - a)^{2n+1}. \tag{15}$$

Since we have $b > a$, we set $b - a = k$, $k \in R^+$. Eliminating y between Eq.(14), and Eq.(15), we obtain

$$cm(x - a)^{2n+1} + m_1(x - a) - m_1 k = 0. \tag{16}$$

In Eq.(16), if we replace

$$Y = x - a \tag{17}$$

then it redues to

$$Y^{2n+1} + \lambda Y - \mu = 0. \tag{18}$$

where $\lambda = m_1/cm$, $\mu = km_1/cm$, $\lambda, \mu \in R^+$. This last equation can not have any negative real roots. According "Descartes' signs rule" and some well known theorems of algebra, we can say that it has only one positive real root. Thus Y is uniqely determined by (18). x and y, also , can be uniqely determined from (17) and (14), respectively.

As a result of Th.2 and Th.3, it can be said that any two different lines, in the plane, intersect at just one point.

Particularly, the family of planes whose lines are defined by expressions (3) and (4), contains the non-desarguesian Moulton's plane for $n = 0$, and the euclidean projective plane for $n = 0$ and $c = 1$.

Furthermore, we can apply this deformation to the upper parts of all straight lines (including the ones with negative slope) of the euclidean plane. The given proofs are also valid for them. But in this case, we need Th. 3 to show that the constructed planes contain the Moulton's plane as a partiular case.

Choosing a point S different from the point

$$(a + c^{-1/2n} \, , \; m \, c^{-1/2n})$$

as the center of perspectivity, from fig. 1, can be easily seen that these planes (except the case $n = 0$, $c = 1$) are non-desarguesian.

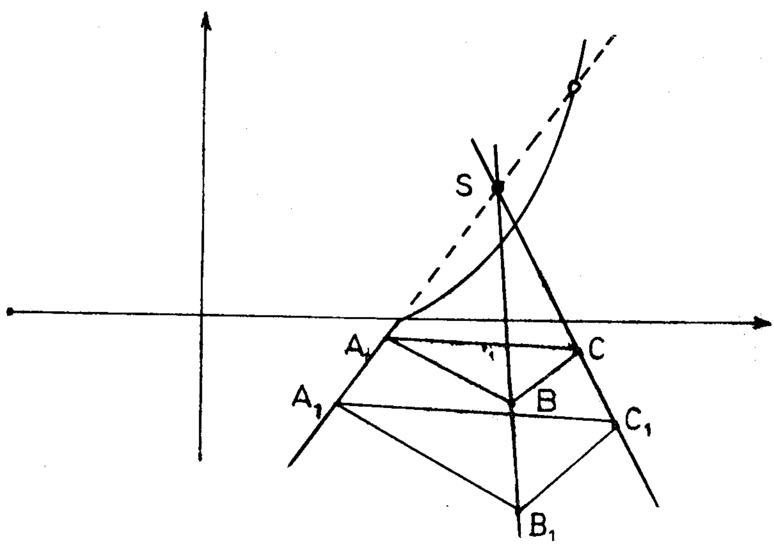

Fig. 1

III. AFFINE GEOMETRY

In this section, we use the following definition, and show that the constructed planar geometries can be regarded as affine geometries at the same time.

Definition. Two lines in a generalized Moulton's plane are called parallel if they do not intersect at a finite point of the plane.

Then we give the following theorem.

Theorem 4. In a generalized Moulton's plane , there is only one line which passes through a given point and parallel to a given line.

It is enough to prove the theorem only in the case when the given point is in the upper half-plane and the given line has positive slope. Let deformed part of the given line be defined by

$$y = cm\,(x - a)^{2n+1} \tag{19}$$

and deformed part of a line passing through the given point (x_1,y_1) be defined by

$$y = cm_1\,(x - a)^{2n+1}. \tag{20}$$

Since the point (x_1, y_1) satisfies the Eq.(20) and the euclidean parts of these lines do not intersect at a finite point, we can have

$$m_1 = m \quad \text{and} \quad b = x_1 - (y_1/cm)^{1/2n+1}.$$

This completes the proof.

REFERENCES

[1] F. R. Moulton, A simple non-desarguesian plane geomrtry, Trans. Amer. Math. Soc., Vol 3, pp.192-195, (1902).

[2] K. Levenberg, A class of non-desarguesian plane geometries, Amer. Math. Monthly, Vol. 57, No. 6, pp.381-387, (1950).

[3] R. Kaya, Construction of a real non-desarguesian plane, Comm. Fac. Sci. Univ. Ankara, Ser. A., No. 2, pp.13-21, (1972).

ÖZET

Bu çalışma ile F. R. Moulton düzlemi [1], K. Levenberg [2] den farklı bir yolla, genelleştirilmektedir. Bunun için reel projektif düzlemin doğrularının üst yarı düzlemde kalan kısımları önce belli bir c oranında kırılır, sonra da tek dereceden ve özel bir şekilde seçilmiş polinom eğrileriyle değiştirilirler. Reel projektif düzlemi de özel hal olarak içine alan bu genelleştirilmiş düzlemler sınıfının her üyesinin bir afin düzlem olarak da göz önüne alınabileceği gösterilir.

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