COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 22 A ANNÉE 1973

On The Cohomology of Categories

by

GEORGES HOFF

13

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

---

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les disciplines scientifiques représentées à la Faculté: Mathématiques pures et appliquées, Astronomie, Physique et Chimie théorique, expérimentale et technique, Géologie, Botanique et Zoologie.

La Revue, à l'exception des tomes I, II, III, comprend trois séries

Série A: Mathématiques, Physique et Astronomie.

Série B: Chimie.

Série C: Sciences naturelles.

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté. Elle accepte cependant, dans la mesure de la place disponible, les communications des auteurs étrangers. Les langues allemande, anglaise et française sont admises indifféremment. Les articles devront etre accompagnés d'un bref sommaire en langue turque.

Adres: Fen Fakültesi Tebliğler Dergisi Fen Fakültesi, Ankara, Turquie.

On The Cohomology of Categories

GEORGES HOFF

INTRODUCTION

We have started our study of the cohomology of categories [1] in particularizing a note of C. Ehresmann [2]. Then, our wish was to put together, in a same work, our original study and the theory of M. André [3]. The result is the text herewith presented.

In the first chapter, we construct a homology and a cohomology of categories. The cohomology is a generalization (different of the one defined in [2]) of the cohomology considered in [1]. The homology is the very same as the one defined in [3].

In the second chapter, we particularize the previous theory in order to obtain a cohomology of the small categories which generalizes the classical cohomology of the groups, a group being a particular category.

These two chapters are written in a different mind; consequently, we have judged useful to differentiate their notations.

We could use also the tools herewith introduced, for example, in a generalization of the study of the groups operating on topological spaces; this will be the subject of an other publication.

Georges HOFF

Universite Paris–Nord,

Centre Scientifique et Polytechnique,

Departement de Mathematiques,

Place du 8 Mai 1945

93206 Saint–Denis FRANCE

Chapter I HOMOLOGY AND COHOMOLOGY OF CATEGORIES

Section 1 FUNCTORS

1.1 *Recalls and notations.*

If **N** is a category (see [4] and [5]), we shall denote by Mor**N** the class of morphisms of **N**. If N is an object of **N**, we shall denote by N the identity morphism of N. If f is a morphism of **N**, we shall denote by $\alpha(f)$ and $\beta(f)$ the domain and the codomain of f respectively.

In the following **A** will be an abelian category such that every set of objects of **A** has a coproduct in **A**. If *N* is a category, we shall denote by [**N**, **A**] the abelian category of natural transformations between functors from **N** to **A**.

We shall consider a category **N** and a full subcategory **M** of **N**. If U is a functor with domain **N**, we shall denote by U |**M** the restriction of U to **M**.

1.2 *Free functors.*

Let T: **M** $\longrightarrow$ **A** be a functor.

Definition: A functor U: **N** $\longrightarrow$ **A** is said T–*free* if for every object N of **N** the object U (N) of **A** is a coproduct of objects of the form T(M) where M is an object of **M**.

Let Σ be a set, n and m applications from Σ to the class of objects of **N** and to the class of objects of **M** respectively. We denote by **N**xΣ the class of pairs $(f,\sigma) \in$ Mor**N**xΣ such that

$$\alpha(f) = n(\sigma)$$

and for each object N of **N** we denote by (**N**xΣ)(N) the set of pairs $(f,\sigma) \in$ **N**xΣ such that

$$\beta(f) = N.$$

We define $<f, \sigma> = T\,(m(\sigma)\,)$ for each $(f, \sigma) \in$ **N**xΣ. For each object N of **N** let U (N) be the coproduct $\coprod_{(f,\ \sigma)\ \varepsilon\ (N \times \Sigma)(N)}$

$<f, \sigma>$. For each morphism g of **N** let U (g) be the morphism from $U(\alpha(g))$ to $U(\beta(g))$ defined by the identity morphisms from $<f, \sigma> = T(m(\sigma))$ to $<g. f, \sigma> = T(m(\sigma))$. Thus we obtain a functor U: **N** ⟶ **A** and, by construction, this functor is T–free.

Definition: This functor U is called the T– *free functor generated* by (Σ, n, m).

(Recall that if for every summand of a coproduct there is given a morphism to some summand of an other coproduct, then this define a morphism from the first coproduct to the second).

Section 2 RESOLUTIONS

2.1 *Free resolutions.*

Let T: **M** ⟶ **A** be a functor.

For each integer $k \geq 0$ let U_k: **N** ⟶ **A** be a T–free functor; for each integer $k > 0$ let ∂_k be a natural transformation from U_k to U_{k-1}; and let ∂_o be a natural transformation from $U_o|$**M** to T.

Definition: $L = (U_k, \partial_k)_k \geq O$ is a T– *free resolution of* T if the sequence

$$\ldots \longrightarrow U_k \xrightarrow{\partial_k} U_{k-1} \longrightarrow \ldots \longrightarrow U_1 \xrightarrow{\partial_1} U_o \longrightarrow 0$$

is a complex of [**N**,**A**] and if for each object M of **M** we have a resolution

$$\ldots \longrightarrow U_k(M) \xrightarrow{\partial_k(M)} U_{k-1}(M) \longrightarrow \ldots \longrightarrow U_o(M) \xrightarrow{\partial_o(M)} T(M) \longrightarrow 0$$

of T (M) in **A** (i.e. an exact complex of **A** over T (M)).

2.2 *Canonical resolutions.*

In the following, we shall consider, as in [3], a category **N** and a full and small subcategory **M** of **N**. Let T: **M** ⟶ **A** be a functor.

Definition: If $k > O$ is an integer, a k–*simplex* of **M** is a sequence $[f_k, \ldots, f_1]$ such that

$$f_i \in \mathrm{Mor}\mathbf{M} \qquad \forall\ i=1,\ldots,k,$$
$$\alpha(f_{i+1}) = \beta(f_i) \qquad \forall\ i=1,\ldots,k-1.$$

Let Σ_k be the set of k-simplexes of **M**, n_k and m_k the applications from Σ_k to the class of objects of **N** and to the set of objects of **M** respectively defined by

$$n_k\colon [f_k,\ldots,f_1] \longrightarrow \beta(f_k),$$
$$m_k\colon [f_k,\ldots,f_1] \longrightarrow \alpha(f_1).$$

Let Σ_o be the set of objects of **M**, n_o and m_o the applications from Σ_o to the class of objects of **N** and to the set of objects of **M** defined by the identity application of Σ_o.

For each integer $k \geq 0$, let U_k: **N**$\longrightarrow$**A** be the T–free functor generated by (Σ_k, n_k, m_k). For each integer $k > 1$, let $\partial^i{}_k$ be the natural transformation from U_k to U_{k-1} defined by the identity morphisms from $< f, [f_k,\ldots,f_1] >$ to $<f.\, f_k, [f_{k-1},\ldots,f_1] >$ for i=k, to $<f, [f_k,\ldots,f_{i+1}.f_i,\ldots,f_1] >$ for $i=1,\ldots,k-1$, and by the morphisms T (f_1) from $<f, [f_k,\ldots,f_1] >$ to $<f, [f_k,\ldots, f_2] >$ for i=O; then we have, for each integer $k > 1$, a natural transformation $\partial_k = \sum_{i=0}^{k} (-1)^{k-i}\partial^i{}_k$ from U_k to U_{k-1}. Let $\partial^i{}_1$ be the natural transformation from U_1 to U_o defined by the identity morphisms from $<f, [f_1] >$ to $<f.\, f_1, \alpha(f_1) >$ for i=1 and by the morphisms T (f_1) from $< f, [f_1] >$ to $<f, \alpha(f) >$ for i=O; then we have a natural transformation $\partial_1 = \partial^1{}_1 - \partial^\circ{}_1$ from U_1 to U_o. Moreover, let ∂_o be the natural transformation from $U_o | M$ to T defined by the morphisms T(f) from $< f, \alpha(f) >$ to T $(\beta(f))$.

Theorem: $L = (U_k, \partial_k)_k \geq 0$ *is a* T*– free resolution of* T.

Proof: A formal simplicial calculus permits to prove that

$$\partial_k . \partial_{k+1} = O \qquad \forall\ k \geq 1,$$

and thus that the sequence

$$\ldots \longrightarrow U_k \xrightarrow{\partial_k} U_{k-1} \longrightarrow \ldots \longrightarrow U_1 \xrightarrow{\partial_1} U_o \longrightarrow O$$

is a complex of [**N**, **A**]. In order to prove that, for each object M of **M**, the augmented complex

$$\ldots \longrightarrow U_k(M) \xrightarrow{\partial_k (M)} U_{k-1}(M) \longrightarrow \ldots$$

$$\longrightarrow U_o(M) \xrightarrow{\partial_o (M)} T\ (M) \longrightarrow 0$$

is a resolution of T (M) in A, we construct morphisms

$$s_k\colon U_k\ (M) \longrightarrow U_{k+1}\ (M)$$

such that

$$\partial_{k+1}\ (M).s_k + s_{k-1}.\partial_k\ (M) = U_k\ (M)$$

(i.e. a contracting homotopy in the sense of [6] p. 41). We define s_k by the identity morphisms from $< f, \alpha (f) >$ to $< \beta (f), [f, \alpha(f)] >$ for $k=0$ and from $< f, [f_k, \ldots, f_1] >$ to $<\beta (f), [f, f_k, \ldots, f_1] >$ for $k > 0$ when f has M for codomain.

Definition: A k–simplex $[f_k, \ldots, f_1]$ of **M** is said *degenerated* if one of the f_i, $i=1,\ldots,k$, is an identity morphism.

For each integer $k > 0$, let Σ°_k be the set of degenerated k–simplexes of **M**, $n_k{}^\circ$ and $m_k{}^\circ$ the restrictions to Σ°_k of the applications n_k and m_k respectively.

For each integer $k > 0$, let U°_k: **N**$\longrightarrow$**A** be the T–free functor generated by $(\Sigma^\circ_k, n_k{}^\circ, m_k{}^\circ)$. In the category [**N**,**A**], the object U°_k is a direct summand of U_k; let $\bar{U}_k$ be the quotient U_k/U°_k. Let $\bar{U}_o$ be U_o.

Theorem: *There exists a* T– *free resolution* $\bar{L}=(\bar{U}_k, \bar{\partial}_k)_k \geq 0$ *of* T *which is a quotient of the* T*–free resolution* L *of* T, *i.e. such that* $\bar{\partial}_k$ *is a quotient morphism of* ∂_k *in the category* [**N**, **A**].

Proof: The question is to prove that, for each integer $k > 0$, the morphosm ∂°_k, defined on the summand U°_k of U_k by ∂_k, factors through the summand U°_{k-1} of U_{k-1}.

Definition: The T–free resolution L (resp. $\bar{L}$) of T is called the *canonical* T*–resolution* (resp. the *canonical normalized* T*–resolution*) of the category **N**.

Section 3 HOMOLOGY AND COHOMOLOGY OF CATEGORIES

3.1 *Definitions.*

Let **N** be a category, **M** a full and small subcategory of **N** and T: **M**$\longrightarrow$**A** a functor.

Let $L=(U_k, \partial_k)_k \geq 0$ be the canonical T–resolution and $\bar{L}=(\bar{U}_k, \bar{\partial}_k)_k \geq 0$ the canonical normalized T–resolution of **N**.

Let $(H_k(\ ,T) \ )_k \geq 0$ and $(\bar{H}_k \ (\ ,T) \ )_k \geq 0$ be the homology graded objects of the complexes of [**N**, **A**] defined by L and $\bar{L}$ respectively.

Definition: For each integer $k \geq 0$, the functor H_k (,T) (resp. $\bar{H}_k$ (,T)) is called the T–*homology functor* (resp. the *normalized* T–*homology functor*) *of order* k from **N** to **A**.

Remarks: The functor H_o (,T) is the Kan extension of T. For every object M of **M** we have

$$H_k(M,T)=0 \ , \ \bar{H}_k(M,T)=0 \qquad \forall \ k > 0 \ .$$

For each integer $k \geq 0$, we denote by $H^k{}_T$ (resp. $\bar{H}^k{}_T$) the functor from [**N**, **A**] to *ab*, the category of abelian groups, such that for each functor U: **N**$\longrightarrow$**A** the group $H^k{}_T$ (U) (resp. $\bar{H}^k{}_T$(U)) is the k–th. cohomology group of the complex of [**N**,**A**] defined by L (resp.$\bar{L}$) with coefficient U.

Definition: The functor $H^k{}_T$ (resp. $\bar{H}^k{}_T$) is called the T–*cohomology functor* (resp. the *normalized* T–*cohomology functor*) *of order* k from [**N**, **A**] to *ab*. The group $H^k{}_T$ (U) (resp. $\bar{H}^k{}_T$ (U)) is called the T–*cohomology group* (resp. the *normalized* T–*cohomology group*) *of order* k of **N** to U.

3.2 *Dimensions.*

To each homology or cohomology defined above corresponds a dimension. It is the greatest integer k, if it exists, such that the corresponding functor is not the zero functor; if such an integer k does not exist, we say that the corresponding dimension of **N** is is infinite.

Definition: If there exists an integer $p \geq 1$ such that 1° for all integer $q > 0$ every $(p+q)$-simplex of *M* is degenerated, 2° there exists a p-simplex of **M** not degenerated, then we say that **M** has the *simplicial dimension* p. If for all integer $k \geq 1$ every k-simplex of **M** is degenerated, i.e. if **M** is a discrete category, then we say thet *M* has the simplicial dimension 0. If for each integer $k \geq 1$ there exists a not degenerated k-simplex of *M*, then we say that **M** has an infinite simplicial dimension.

Remark: If *M* has the simplicial dimension $p \geq 1$, for each integer k such that

$$1 \leq k \leq p ,$$

there exists a non degenerated k-simplex of **M**.

Examples: The category 2, with only one morphism, the domain and codomain of which are distincts, has the simplicial dimension 1. A not discrete category with only one object has an infinite simplicial dimension, in particular it is the case for the groups.

This notion of simplicial dimension permits to majore certain of the dimensions of **N**.

Theorem: *Let* **N** *be a category and* **M** *a full and small subcategory of* **N**. *If the category* **M** *has the simplicial dimension* $p \geq 0$, *then for all functor* $T: \mathbf{M} \longrightarrow \mathbf{A}$ *and for each integer* $k > p$, *the functors* $\overline{H}^k_T(\ ,T)$ *and* $\overline{H}^k_T$ *are the zero functors from* **N** *to* **A** *and from* [**N**, **A**] *to ab respectively.*

Proof: For each integer $k > p$, the set Σ°_k of degenerated k-simplexes of *M* is the set Σ_k of all k-simplexes of **M**. Then the subobject U°_k of U_k is U_k; consequently, the quotient $\bar{U}_k$ is null. It follows that the normalized homology and cohomology functors of order k are the zero functors.

CONCLUSION

The homology obtained here is the one defined by M. André in [3]. S. Swierczkowski has shown in [7] that it is a particular case of Barr and Beck homology defined with cotriple notion.

We do not give the properties of homology and cohomology functors defined in this chapter. The methods are those of [8] and for the properties of homology functor, see [3].

In the following chapter, we shall consider the case where the categories N and M are the same, where A is the category *ab* of abelian groups and where the functor T is defined by the constant functor T: M ——→Z, Z being the group of integers. Then we shall consider only cohomology, because the defined homology is null. The canonical T-resolutions being projective resolutions, the cohomologies (normalized or not) coincide.

In order to point out that the obtained cohomology is a generalization to the small categories of the classical cohomology of groups, we shall use different notations.

Chapter II A COHOMOLOGY OF SMALL CATEGORIES

Section 1 MODULES ON A CATEGORY

1.1 *Recalls and notations.*

In this chapter, all categories are small categories.

We shall denote by $C^{\cdot}$ a category where the set of elements (morphisms) is C and where the law of composition is“.”.

The set of units of a category $C^{\cdot}$ (which is in 1 to 1 correspondence with the set of objects of this category) will be denoted by $C^{\cdot}_0$. We shall denote by α and β the applications from C to $C^{\cdot}_0$ which associate to each morphism $f \in C$ its domain and its codomain respectively.

If $C^{\cdot}$ is a category and if e and e′ are two units of $C^{\cdot}$ then we shall denote by $C^{\cdot}(e,e')$ the set of morphisms $f \in C$ with e as domain and e′ as codomain.

Recall that a groupoid is a category in which all morphism is invertible; a group is a groupoid with only one unit (the unit element of the group).

1.2 *Modules on a category.*

Let *ab* be the category of abelian groups. Let $C^\cdot$ be a category.

Definition: A $C^\cdot$ *-module* is a functor $M: C^\cdot \longrightarrow ab$ such *that*

$$M(e) \cap M(e') = \varnothing \qquad \forall e \in C^\cdot_0, \forall e' \in C^\cdot_0, e \neq e' ;$$

a natural transformation φ between such two functors is called $C^\cdot$ -linear application.

Remarks: A $C^\cdot$ -module M defines an "espèce de structures ($C^\cdot$, M) dominée par les groupes abéliens" and a $C^\cdot$ -linear application φ defines an "application covariante ($C^\cdot$, φ) entre espèces de structures dominées par les groupes abéliens"; these notions are defined in [4]. We may represent a $C^\cdot$ -module as a fibration of $C^\cdot_0$ with abelian groups as "fibers", each morphism $f \in C$ defines a group homomorphism from the "fiber" over its domain to the "fiber" over its codomain. Any functor from $C^\cdot$ to *ab* defines a $C^\cdot$ -module, "separating the fibers", if necessary, by an indexation of images of units.

The $C^\cdot$ -linear applications between $C^\cdot$ -modules define a full subcategory of the category of natural transformations between functors from $C^\cdot$ to *ab*. This category is abelian and will be denoted by Mod $C^\cdot$.

Definition: A *free* $C^\cdot$ -module is a $C^\cdot$ -module M such that for each $e \in C^\cdot_0$ the group M (e) is a free abelian group.

Remark: The free $C^\cdot$ -modules are defined by the **Z** -free functors, in the sense of chapter I, $\mathbf{Z}: C^\cdot \longrightarrow ab$ being the $C^\cdot$ -module defined by

$$\mathbf{Z}(e) = \{e\} \times \mathbf{Z} \qquad \forall e \in C^\cdot_0 ,$$

$$\mathbf{Z}(f)(e,p) = (e',p) \qquad \forall f \in C^\cdot(e, e') ,$$

where **Z** is the group of integers.

If M is a $C^\cdot$ -module, for each $e \in C^\cdot_0$ we shall denote additively the law of composition in the group M (e) and its unit will be denoted by $0_{M(e)}$. If $f \in C$ is a morphism, for each $z \in M(\alpha(f))$ we shall write fz the image M (f) (z) of z by the group homomorphism M (f).

We shall see, at the end of this chapter, that this notion of module on a category is compatible with those of generalized module on a predditive category (i. e. on a ringoid).

Section 2 CROSSED HOMOMORPHIMS

Let $C^{\cdot}$ be a category.

2.1 *Crossed product.*

Let M be a $C^{\cdot}$ –module; by definition, the category $C^{\cdot}$ operates on the groupoid sum of groups $\sum_{e \varepsilon C^{\cdot}_0} M(e)$ with the law of composition $\varkappa$ defined by

$$\varkappa (f, z) = fz$$

if and only if fz is defined, i.e. for $z \varepsilon M(\alpha(f))$.

Let $M x_{\varkappa} C^{\cdot}$ be the set of pairs (z, f) such that

$$f \varepsilon C, z \varepsilon M(\beta(f)) .$$

Let $(M x_{\varkappa} C^{\cdot})^{+}$ be the set $M x_{\varkappa} C^{\cdot}$ together with the law of composition defined by

$$(z', f') + (z, f) = (z' + f' z, f'.f)$$

if and only if we have

$$\alpha(f') = \beta(f) .$$

Proposition: $(M x_{\varkappa} C^{\cdot})^{+}$ *is a category and* $\sum_{e \varepsilon C^{\cdot}_0} M(e)$ *is identified to a subcategory of* $(M x_{\varkappa} C^{\cdot})^{+}$. *Furthermore* $C^{\cdot}$ *is a quotient category of*⁻ $(M x_{\varkappa} C^{\cdot})^{+}$, *the projection functor being defined by*

$$\pi: (z,f) \longrightarrow f .$$

Proof: Clearly $(M x_{\varkappa} C^{\cdot})^{+}$ is a category whose units are pairs $(O_{M(e)}, e)$ for all $e \varepsilon C^{\cdot}_0$. The functor $\iota: \sum_{e \varepsilon C^{\cdot}_0} M(e) \longrightarrow (M x_{\varkappa} C')^{+}$ which sends each $z \varepsilon M(e)$ onto $(z, e) \varepsilon M x_{\varkappa} C^{\cdot}$ gives an identification of $\sum_{e \varepsilon C^{\cdot}_0} M(e)$ to the subcategory of $(M x_{\varkappa} C^{\cdot})^{+}$ constituted by the (z,e) for all $e \in C^{\cdot}_{o}$. Let ρ be the equivalence relation defined on $M x_{\varkappa} C^{\cdot}$ by

$$(z,f) \sim (z',f') \quad \Leftrightarrow \quad f=f';$$

it is compatible with the law of composition of $(M x_{\varkappa} C^{\cdot})^{+}$. The functor π: $(M x_{\varkappa} C^{\cdot})^{+} \longrightarrow C^{\cdot}$ defines an 1 to 1 correspondence between the quotient $M x_{\varkappa} C^{\cdot} / \rho$ and the set C; The law of composition in C defined by the one of $(M x_{\varkappa} C^{\cdot})^{+}$ and this 1 to 1 correspondence is the law of $C^{\cdot}$; the class modulo ρ of all unit $(O_{M(e)}, e)$ of $(M x_{\varkappa} C^{\cdot})^{+}$ corresponds to the unit e of $C^{\cdot}$. Thus the category $C^{\cdot}$ is a quotient category of $(M x_{\varkappa} C^{\cdot})^{+}$, the projection functor being the functor π.

The units of $(M x_{\varkappa} C^{\cdot})^{+}$ are identified to the units of its subcategory $\sum_{e \in C^{\cdot}_{o}} M(e)$.

Definition: The category $(M x_{\varkappa} C^{\cdot})^{+}$ is called the *crossed product* of M and $C^{\cdot}$.

2.2 *Crossed homomorphisms.*

Let M be a $C^{\cdot}$ –module.

Definition: A *crossed homomorphism* of $C^{\cdot}$ to M is an application ψ from C to $\sum_{e \in C^{\cdot}_{o}} M(e)$ such that

$$\psi(f) \in M(\beta(f)) \qquad \forall f \in C,$$

$$\psi(f'.f) = \psi(f') + f'\psi(f) \qquad \text{if } \alpha(f') = \beta(f).$$

Remark: If ψ is a crossed homomorphism, we have

$$\psi(e) = O_{M(e)} \qquad \forall e \in C^{\cdot}_{o}$$

The sum of two crossed homomorphisms ψ_1 and ψ_2 defined by

$$(\psi_1 + \psi_2)(f) = \psi_1(f) + \psi_2(f) \qquad \forall f \in C,$$

is a crossed homomorphism. Under this law of composition the set of crossed homomorphisms of $C^{\cdot}$ to M is an abelian group, which will be denoted by $Z^1(M)$ and whose unit is the crossed homomorphism ψ_o defined by

$$\psi_o(f) = O_{M(\beta(f))} \qquad \forall f \in C.$$

Let τ be an application from $C^{\cdot}_{o}$ to $\sum_{e \in C^{\cdot}_{o}} M(e)$ such that

$$\tau(e) \in M(e) \qquad \forall e \in C^{\cdot}_{o}.$$

Let ψ_τ be the application from C to $\sum_{e \in C^{\cdot}_{o}} M(e)$ defined by it

$$\psi_\tau(f) = \tau(e') - f\tau(e) \qquad \forall f \in C^{\cdot}(e, e');$$

it is a crossed homomorphism of $C^{\cdot}$ to M.

Definition: A crossed homomorphism of the form ψ_τ is called *principal crossed homomorphism.*

Since

$$\psi_{\tau_1} + \psi_{\tau_2} = \psi_{(\tau_1+\tau_2)},$$

where $(\tau_1 + \tau_2)$ is the application from $C^{\cdot}_{o}$ to $\sum_{e \in C^{\cdot}_{o}} M(e)$ such that

$$(\tau_1 + \tau_2)(e) = \tau_1(e) + \tau_2(e) \qquad \forall e \in C^{\cdot}_{o},$$

the principal crossed homomorphisms constitute a subgroup $B^1(M)$ of $Z^1(M)$.

If τ is an application from $C^{\cdot}_{o}$ to $\sum_{e \in C^{\cdot}_{o}} M(e)$ such that the

$$\tau(e) \in M(e) \qquad \forall e \in C^{\cdot}_{o},$$

the "inner automorphism" of $(M \times_{\chi} C^{\cdot})^{+}$ defined by τ is the functor from the crossed product to itself defined by

$$(z, f) \longrightarrow (\tau(e), e) + (z, f) + (-\tau(e'), e')$$
$$= (\tau(e) + z - f\tau(e'), f),$$

where

$$e' = \alpha(f), \; e = \beta(f).$$

Proposition: The group of all those invertible functors from the crossed product to itself which induce the identity functors both on the subcategory $\sum_{e \in C^{\cdot}_{o}} M(e)$ *and the quotient category* $C^{\cdot}$ *is isomorphic to the group* $Z^1(M)$ *of crossed homomorphisms. Under this isomorphism the inner automotrphisms of* $(M \times_{\chi} C^{\cdot})^{+}$ *defined by the applications* τ *from* $C^{\cdot}_{o}$ *to* $\sum_{e \in C^{\cdot}_{o}} M(e)$ *such that*

$$\tau(e) \in M(e) \qquad \forall\, e \in C^{\cdot}_{o}$$

correspond to the principal crossed homomorphisms.

Proof: A functor as described must be given by a formula

$$\Psi^{\cdot}(z, f) = (z + \psi(f), f) \qquad \forall\, (z, f) \in M x_{\varkappa} C^{\cdot}\ ,$$

for some application ψ from C to $\sum_{e \in C^{\cdot}_{o}} M(e)$ such that

$$\psi(f) \in M(\beta(f)) \qquad \forall\, f \in C\ ,$$

$$\psi(e) = O_{M}(e) \qquad \forall\, e \in C^{\cdot}_{o}\ .$$

The fact that ψ does not depend of z is an immediate consequence of the fact that $\Psi^{\cdot}$ must be a functor and that we must obtain

$$\Psi^{\cdot}(z,f) = \Psi^{\cdot}(z, \beta(f)) + \Psi^{\cdot}(O_{M}(\beta(f)), f).$$

Since $\Psi^{\cdot}$ must be a functor, we must have the following relation

$$\Psi^{\cdot}((z^{\cdot}, f^{\cdot}) + (z,f)) = \Psi^{\cdot}(z^{\cdot}, f^{\cdot}) + \Psi^{\cdot}(z, f) \qquad \text{if } \beta(f) = \alpha(f^{\cdot})$$

which is equivalent to the following

$$\Psi^{\cdot}(f^{\cdot}.f) = \psi(f^{\cdot}) + f^{\cdot}\psi(f) \qquad \text{if } \beta(f) = \alpha(f^{\cdot}).$$

Conversely a such defined functor $\Psi^{\cdot}$ is invertible, because the application $(-\psi)$ from C to $\sum_{e \in C^{\cdot}_{o}} M(e)$ defined by

$$(-\psi)(f) = -\psi(f) \qquad \forall\, f \in C$$

defines a functor $\Psi^{\cdot -1}$ by the formula

$$\Psi^{\cdot -1}(z, f) = (z + (-\psi)(f), f) \qquad \forall\, (z,f) \in M x_{\varkappa} C^{\cdot}\ ,$$

and $\Psi^{\cdot -1}$ is an inverse of $\Psi^{\cdot}$. The composition of the functors $\Psi^{\cdot}$ corresponds to the sum of the applications ψ and the inner automorphisms of $(M x_{\varkappa} C^{\cdot})^{+}$ defined by the applications τ correspond to the principal crossed homomorphisms.

2.3 *The cohomology group of order* 1.

Let M be a $C^{\cdot}$ –module. We shall denote by $H^1(M)$ the quotient of abelian groups $Z^1(M)/B^1(M)$.

Definition: The abelian group $H^1(M)$ is called the *cohomology group of order* 1 *of* $C^{\cdot}$ *to* M.

Let M and $\bar{M}$ be two $C^{\cdot}$ –modules, if φ is a $C^{\cdot}$ –linear application from M to $\bar{M}$ and if ψ is a crossed homomorphism of $C^{\cdot}$ to M, the formula

$$[Z^1(\varphi)\,\psi](f) = \varphi\,(M(\beta(f)))\,(\psi(f)) \qquad \forall\, f \in C$$

gives a group homomorphism $Z^1(\varphi)$ from $Z^1(M)$ to $Z^1(\bar{M})$ which makes Z^1 and H^1 functors from Mod $C^{\cdot}$ to *ab*.

Remark: These functors can be extended to functors from the "catégorie des applications contravariantes entre espèces de structures dominées par les groupes abéliens" (see [4]) to *ab*.

Nota: The notions of crossed product and crossed homomorphism considerated in this section are particularizations of the ones defined in [9] together with the notion of cohomology classes of order 1 of a "catégorie munie d'une catégorie d'opérateurs".

Section 3 EXTENSIONS

3.1 *Extensions of categories.*

Definition: An *extension of categories* is a sequence

$$E:\ H^+ \xrightarrow{\ \iota\ } K^+ \xrightarrow{\ \pi\ } C^{\cdot}$$

such that the functor ι identifies H^+ to a subcategory of K^+ and $C^{\cdot}$ is a quotient category of K^+, the projection functor being the functor π; furthermore we require that for all $k_1 \in K$ and $k_2 \in K$

$$\pi(k_1) = \pi(k_2) \Leftrightarrow \exists!\, z \in H \text{ such that } k_2 = \iota(z) + k_1 .$$

Remarks: If E is an extension of categories we have

i) $\alpha(\pi(k_2)) = \beta(\pi(k_1)) \Rightarrow \alpha(k_2) = \beta(k_1)$;

ii) $\pi\iota(H) \subset C'_0$;

iii) For every $k \in K$ and $z \in H$ such that

$$\beta(\iota(z)) = \alpha(k)$$

there exists an unique $z_1 \in H$ such that

$$k + \iota(z) = \iota(z_1) + k \; ; \qquad (1)$$

and we shall write

$$\varkappa_E (k, z) = z_1 .$$

Let E and Ê be two extensions of categories.

Definition: A *morphism of extensions* from E to Ê is a triple $\Gamma = (\eta, \mu, \upsilon)$ of functors such that the following diagram is commutative

$$\begin{array}{cccccc} E: & H^+ & \xrightarrow{\iota} & K^+ & \xrightarrow{\pi} & C^{\cdot} \\ & \downarrow \eta & & \downarrow \mu & & \downarrow \upsilon \\ \hat{E}: & \hat{H}^+ & \xrightarrow{\hat{\iota}} & \hat{K}^+ & \xrightarrow{\hat{\pi}} & \hat{C}^{\cdot}. \end{array}$$

Obviously, the composition of morphisms of extensions may be defined. A morphism of extensions $\Gamma = (\eta, \mu, \upsilon)$ is called isomorphism of extension if the functors η, μ and υ are invertible.

3.2 *Extensions of modules on a category.*

Let $C^{\cdot}$ be a category and M a $C^{\cdot}$ –module.

Definition: An *extension of* M *by* $C^{\cdot}$ is an extension of categories

$$E: \sum_{e \varepsilon C^{\cdot}_0} M (e) \xrightarrow{\iota} K^+ \xrightarrow{\pi} C^{\cdot}$$

such that for all $k \varepsilon K$ the conditions

$$z \varepsilon M (\alpha (\pi (k))) \quad \text{and} \quad \alpha (k) = \beta (\iota (z))$$

are equivalent and that we have then

$$\varkappa_E (k, z) = \pi (k) z .$$

Remark: for every $e \varepsilon C^{\cdot}_0$ and $z \varepsilon \sum_{e \varepsilon C^{\cdot}_0} M (e)$ we have

$$z \varepsilon M (e) \Leftrightarrow \pi \iota (z) = e \tag{2}$$

The problem is that of constructing all extension E of M by $C^{\cdot}$. There is at least one such extension, taking the crossed product $(M x_{\varkappa} C^{\cdot})^+$ as category K^+ and the functors ι and π of the proposition 2.1.

Let E and Ê be two extensions of M by $C^{\cdot}$.

Definition: A *congruence* from E to Ê is an isomorphism of extensions $\Gamma = (\eta, \mu, \upsilon)$ from E to Ê with

$$\eta = \sum_{e \varepsilon C^{\cdot}_{o}} M\ (e) \text{ and } \upsilon = C^{\cdot}\ .$$

A congruence from E to Ê defines obviously a congruence from Ê to E; we may therefore speak of congruence classes of extensions of M by $C^{\cdot}$. Let Ext (M, $C^{\cdot}$) denote the set of all congruence classes of extensions of the $C^{\cdot}$ –module M by $C^{\cdot}$. We wish to describe the set Ext (M, $C^{\cdot}$).

3.3 *Factor application of an extension.*

Let $C^{\cdot}$ be a category and M a $C^{\cdot}$ –module. Let

$$E\colon \sum_{e \varepsilon C^{\cdot}_{o}} M\ (e) \xrightarrow{\iota} K^{+} \xrightarrow{\pi} C^{\cdot}$$

be an extension of M by $C^{\cdot}$. For convenience, identify each $z \ \varepsilon \sum_{e \varepsilon C^{\cdot}_{o}} M\ (e)$ with ι (z).

To each f ε C choose a "representative" u (f) in K; that is an u (f) ε K with π (u(f)) = f. In particular, choose u (e) in K^{+}_{o} for each e ε $C^{\cdot}_{o}$. Now each class modulo π in K contains exactly one u (f) and the elements of K can be represented uniquely as z+u (f) for a $z \ \varepsilon \sum_{e \varepsilon C'_{o}} M\ (e)$ and a f ε C. Then (1), for k=u (f), becomes

$$u\ (f) + z = fz + u\ (f) \qquad \forall z \ \varepsilon\ M\ (\alpha(f))\ . \tag{3}$$

On the other hand, if $C^{\cdot}{}_{*}C^{\cdot}$ is the set of pairs (f_2, f_1) ε CxC such that $f_2 . f_1$ is defined in $C^{\cdot}$, for each (f_2,f_1) ε $C^{\cdot}{}_{*}C^{\cdot}$ the element u (f_2) + u (f_1), which is defined according to the remark i) of 3.1, must lie in the same class modulo π as u ($f_2 . f_1$); so, from the definition of an extension, there is an unique ψ (f_2, f_1) $\varepsilon \sum_{e \varepsilon C^{\cdot}_{o}} M\ (e)$ such that

$$u\ (f_2) + u\ (f_1) = \psi\ (f_2, f_1) + u\ (f_2 . f_1)\ . \tag{4}$$

applying π to (4), and according to (2), we have

$$\psi\ (f_2, f_1)\ \varepsilon\ M\ (\beta\ (f_2)\ ) \qquad \forall (f_2, f_1)\ \varepsilon\ C^{\cdot} * C^{\cdot}.$$

Since for all e ε $C^{\cdot}_{o}$ the element u (e) is in K_{o}^{+}, we also have

$$\psi\ (f_2, e) = O_M\ (\beta(f_2)\ ) \text{ and } \psi\ (e, f_1) = O_{M(e)} \quad \forall e \varepsilon C^{\cdot}_{o}. \tag{5}$$

Definition: The application ψ from $C^{\cdot}{}_{*}C^{\cdot}$ to $\Sigma_{e \varepsilon C^{\cdot}{}_{o}} M (e)$ is called *factor* of the extension E.

By this factor and the $C^{\cdot}$ –module M, the law of composition in K^{+} is determined, the composition of any two elements $z_2 + u (f_2)$ and $z_1 + u (f_1)$ of K can be calculated, if it is defined, by (3) and (4) as

$$(z_2 + u (f_2)) + (z_1 + u (f_1)) = (z_2 + f_2 z_1 + \psi (f_2, f_1)) + u (f_2 . f_1) . \quad (6)$$

By this rule, for each $(f_3, f_2, f_1) \varepsilon C^{\cdot}*C^{\cdot}*C^{\cdot}$, we have

$$(u(f_3) + u(f_2)) + u(f_1) = \psi (f_3, f_2) + \psi (f_3 . f_2, f_1) + u(f_3 . f_2 . f_1),$$

$$u(f_3)+(u(f_2)+u (f_1)) = f_3 \psi (f_2,f_1)+ \psi(f_3,f_2 . f_1)+u(f_3 . f_2 . f_1) ;$$

the equality of these two triple sums (associativity of the law of composition of K^{+}) and the unicity of the element $z \varepsilon \Sigma_{e \varepsilon C^{\cdot}{}_{o}} M (e)$ such that $u (f_3 . f_2 . f_1) = z + u (f_3 . f_2 . f_1)$ give

$$f_3 \psi (f_2, f_1) + \psi (f_3, f_2 . f_1) = \psi (f_3, f_2) + \psi (f_3 . f_2, f_1) \quad \forall (f_3, f_2, f_1) \varepsilon C^{\cdot}{}_{*}C^{\cdot}{}_{*}C^{\cdot} . \quad (7)$$

Remark: If E and Ê are two extensions of M by $C^{\cdot}$ such that there is a congruence between E and Ê, and if ψ is a factor of E, then ψ is a factor of Ê; If E and Ê are two extensions of M by $C^{\cdot}$ having a same application ψ for factor, then there is a congruence between E and Ê.

The factor ψ for an extension depends on a choice of representatives. If u'(f) is a second set of representatives with u' (e) in $K^{+}{}_{o}$ for each $e \varepsilon C^{\cdot}{}_{o}$, then u'(f) and u (f) lie in the same class modulo π, so there is an application τ from C to $\Sigma_{e \varepsilon C^{\cdot}{}_{o}} M (e)$ such that $u'(f) = \tau (f) + u (f)$ for every $f \varepsilon C$. Clearly we have

$$\tau (f) \varepsilon M (\beta (f)) \qquad \forall f \varepsilon C, \tau(e) = O_{M (e)} \quad \forall e \varepsilon C^{\cdot}{}_{o} .$$

Thus for every $(f_2, f_1) \varepsilon C^{\cdot}*C^{\cdot}$ we have

$$u' (f_2) + u' (f_1) = \tau (f_2) + f_2 \tau (f_1) + \psi(f_2,f_1) + u (f_2 . f_1) ,$$

and from the unicity of the element $z \varepsilon \Sigma_{e \varepsilon C^{\cdot}{}_{o}} M (e)$ such that

$u'(f_2) + u'(f_1) = z + u(f_2 . f_1)$, the new factor is such that

$$\psi'(f_2, f_1) = \psi_\tau(f_2, f_1) + \psi(f_2, f_1) \qquad \forall (f_2, f_1) \in C^\cdot{}_* C^\cdot,$$

where ψ_τ is the application from $C^\cdot{}_* C^\cdot$ to $\sum_{e \in C^\cdot{}_o} M(e)$ defined by

$$\psi_\tau(f_2, f_1) = f_2 \tau(f_1) - \tau(f_2 . f_1) + \tau(f_2) \qquad \forall (f_2, f_1) \in C^\cdot * C^\cdot \quad (8)$$

One verifies that ψ_τ does satisfy the identity (7), with ψ replaced by ψ_τ there.

3.4 *The cohomology group of order* 2.

Let $C^\cdot$ be a category and M a $C^\cdot$ –module. Let $Z^2(M)$ denote the set of all applications ψ from $C^\cdot{}_* C^\cdot$ to $\sum_{e \in C^\cdot{}_o} M(e)$ which satisfy the following conditions,

1° $\psi(f_2, f_1) \in M(\beta(f_2)) \qquad \forall (f_2, f_1) \in C^\cdot * C^\cdot$,

2° $\psi(f_2, f_1) = O_M(\beta(f_2)) \qquad$ if $f_i \in C^\cdot{}_o$ for i=1 or 2 ,

3° $f_3 \psi(f_2, f_1) + \psi(f_3, f_2 . f_1) = \psi(f_3, f_2) + \psi(f_3 . f_2, f_1)$

$$\forall (f_3, f_2, f_1) \in C^\cdot * C^\cdot * C^\cdot .$$

This set is an abelian group under the law defined by

$$(\psi + \psi')(f_2, f_1) = \psi(f_2, f_1) + \psi'(f_2, f_1) \qquad \forall (f_2, f_1) \in C^\cdot * C^\cdot .$$

Let $B^2(M)$ denote the subset of $Z^2(M)$ which consists of all applications of the form ψ_τ, defined by (8), where τ is an application from C to $\sum_{e \in C^\cdot{}_o} M(e)$ satisfying the following conditions,

1° $\tau(f) \in M(\beta(f)) \qquad \forall f \in C$,

2° $\tau(e) = O_M(e) \qquad \forall e \in C^\cdot{}_o$;

it is a subgroup of $Z^2(M)$. We shall denote by $H^2(M)$ the quotient of abelian groups $Z^2(M) / B^2(M)$.

Definition: The abelian group $H^2(M)$ is called the *cohomology group of order* 2 *of* $C^\cdot$ *to* M.

Proposition: *The application which assigns to each extension of* M *by* $C^\cdot$ *the equivalence class of its factors defines a* 1 *to* 1 *correspondence*

$$\omega : \text{Ext}(M, C^\cdot) \to H^2(M)$$

between the set of all congruence classes of such extensions and the cohomology group of order 2 *of* C˙ *to* M. *Under this correspondence the crossed product corresponds to the unit of* H^2 (M).

Proof: Since the factor of an extension is well defined modulo the subgroup B^2 (M), and since congruent extensions have the same factor, we know that the application ω is well defined. If two extensions yield factors whose difference is some application $\psi_\tau \varepsilon$ B^2 (M), then a change of representatives in one extension will make the factors equal and the extensions congruent; therefore ω is a 1 to 1 correspondence of Ext (M, C˙) with a part of H^2(M). Given any $\psi \varepsilon Z^2$ (M), one may define a category K^+ to consist of pairs (z, f), where f ε C and zεM (β(f)), with a law of composition given as in (6) by

$$(z_2, f_2) + (z_1, f_1) = (z_2 + f_2 z_1 + \psi (f_2, f_1), f_2 . f_1)$$

when the last term is defined, i.e. for every $(f_2, f_1) \varepsilon C^{\cdot} * C^{\cdot}$; the C˙–module rules and the above condition 3° show that this composiposition is associative; it clearly yields an extension of M by C˙

$$\iota (z) = (z,e) \quad \forall z \varepsilon M (e), \qquad \pi(z,f) = f \quad \forall (z,f) \varepsilon K,$$

with representatives u (f) = $(O_M(\beta(f)), f)$ and factor ψ. Then ω is a 1 to 1 correspondence. Finally, observe that the crossed product $(M x_\chi C^{\cdot})^+$ has the trivial application

$$\psi (f_2, f_1) = O_M (\beta(f_2)) \qquad \forall (f_2, f_1) \varepsilon C^{\cdot}{}_* C^{\cdot}$$

as one of its factors.

Section 4 COHOMOLOGY OF CATEGORIES

4.1 *The cohomology functor.*

Let C˙ be a category, Mod C˙ the category of C˙ –modules.

Let *Z* the C˙ –module such that

$$Z (e) = \{e\} \times Z \qquad \forall e \varepsilon C^{\cdot}_o ,$$

$$Z (f) (e,p) = (e',p) \qquad \forall f \varepsilon C^{\cdot} (e,e').$$

The canonical normalized *Z* –resolution of C˙, in the sense of chapter I, defines a projective resolution $L = (M_k, \partial_k)_{k \geq 0}$ of *Z* in Mod $C^\cdot$. For each integer $k \geq 0$, let H^k denote the functor from Mod $C^\cdot$ to *ab* such that, for every $C^\cdot$ –module M, the group $H^k(M)$ is the k-th cohomology group of the complex L with coefficient M. This functor is defined by the cohomology functor and by the *normalized cohomology functor* of chapter I (these are equal because they are defined by two projective resolutions of a same object in Mod $C^\cdot$).

Definition: H^k is called the *cohomology functor of order* k from Mod $C^\cdot$ to *ab* and H^k (M) the *cohomology group of order* k of $C^\cdot$ to M.

Remark: This functor can be extended to a functor from the "catégorie des applications contravariantes entre espèces de structures dominées par les groupes abéliens" to *ab*.

4.2. *Cohomology of order* 1 *and* 2.

Let $C^\cdot$ be a category and $L = (M_k, \partial_k)_{k} \geq 0$ the resolution of Z in Mod $C^\cdot$ defined by the canonical normalized Z –resolution of $C^\cdot$, in the sense of chapter I, and let M be a $C^\cdot$ –module.

For each $e \in C^\cdot_0$, the group $M_k(e)$ is the free abelian group generated by all the pairs $(f, \sigma) \in C \times (\Sigma_k - \Sigma^\circ_k)$ for $k > 0$, $\in C \times C^\cdot_0$ for $k=0$, such that

$$\alpha (f) = n_k (\sigma), \ \beta(f) = e \ ;$$

the such pairs of the form (e, σ) are identified to σ ; for each $g \in C$, the group homomorphism $M_k(g)$ from $M_k(\alpha(g))$ to $M_k(\beta(g))$ is defined by the application

$$(f, \sigma) \longrightarrow (g.f, \sigma).$$

The $C^\cdot$ –linear application ∂_k is determined by the application

$$\begin{aligned}[f_k,\ldots,f_1] \to f_k [f_{k-1},\ldots,f_1] & \\ + \sum_{i<k} (-1)^{k-i} [f_k,\ldots,f_{i+1}.f_i,\ldots,f_1] & \\ + (-1)^k [f_k,\ldots,f_2] &\end{aligned}$$

for $k > 1$, by the application

$$[f] \to f\, \alpha (f) - \beta (f)$$

for $k=1$ and by the application

$$e \rightarrow (e, 1)$$

for $k=O$. All degenerated simplex being identified to the unit of the corresponding group M_k (e).

Then a k–cochain φ from L to M is determined by an application $\bar{\varphi}$ from Σ_k to $\Sigma_{e \varepsilon C^{\cdot}_o}$ M (e) such that

$$\bar{\varphi}([f_k,\ldots,f_1])\varepsilon M(\beta(f_k)) \qquad \forall [f_k,\ldots,f_1]\varepsilon\ \Sigma_k,$$

$$\bar{\varphi}([f_k,\ldots,f_1])=O_{M(\beta(f_k))} \qquad \forall [f_k,\ldots,f_1]\varepsilon\ \Sigma^{\circ}_k$$

Thus a 2–cochain φ from L to M is defined by an application $\bar{\varphi}$ from $C^{\cdot}{}_{*}C^{\cdot}$ to $\Sigma_{e \varepsilon C^{\cdot}_o}$ M (e) such that

$$\bar{\varphi}(f_2,f_1)\ \varepsilon M(\beta(f_2)) \qquad \forall (f_2,f_1)\ \varepsilon\ C^{\cdot}*C^{\cdot}\ ,$$

$$\bar{\varphi}(f_2,f_1)\ =\ O_{M(\beta(f_2))} \qquad \text{if } f_i\ \varepsilon C^{\cdot}_o \text{ for } i=1 \text{ or } 2\ ;$$

the fact that φ is a 2–cocycle is equivalent to

$$f_3\ \bar{\varphi}\ (f_2,f_1) + \bar{\varphi}\ (f_3,\ f_2.f_1) = \bar{\varphi}\ (f_3,\ f_2) + \bar{\varphi}\ (f_3.f_2,\ f_1)$$

$$\forall\ (f_3,\ f_2,\ f_1)\ \varepsilon\ C^{\cdot}*C^{\cdot}*C^{\cdot}.$$

A 1–cochain φ from L to M is defined by an application $\bar{\varphi}$ from C to $\Sigma_{e \varepsilon C^{\cdot}_o}$ M (e) such that

$$\bar{\varphi}(f)\ \varepsilon\ M\ (\beta(f)) \qquad \forall\ f\ \varepsilon\ C\ ,$$

$$\bar{\varphi}\ (e) = O_{M(e)} \qquad \forall\ e\ \varepsilon\ C^{\cdot}_o\ .$$

The coboundary $\delta^1\varphi$ of a 1–cochain φ is given by the formula

$$\overline{\delta^1\varphi}\ (f_2,\ f_1) = f_2\bar{\varphi}(f_1) - \bar{\varphi}\ (f_2.f_1) + \bar{\varphi}(f_2) \qquad \forall (f_2,f_1)\varepsilon C^{\cdot}{}_{*}C^{\cdot}.$$

Consequently, *the cohomology group of order* 2 *of* $C^{\cdot}$ *to* M *defined in* 4.1 *is the one constructed in the section* 3.

The fact that a 1–cochain φ is a 1–cocycle is equivalent to

$$\bar{\varphi}(f_2.f_1)\ =\ \bar{\varphi}(f_2) + f_2\ \bar{\varphi}\ (f_1)\ \forall (f_2,f_1)\ \varepsilon\ C^{\cdot}*C^{\cdot},$$

i.e. to the fact that $\bar{\varphi}$ is a crossed homomorphism of $C^{\cdot}$ to M.

A O–cochain φ from L to M is defined by an application $\bar{\varphi}$ from $C^{\cdot}_{o}$ to $\sum_{e \varepsilon C^{\cdot}_{o}} M(e)$ such that

$$\bar{\varphi}(e) \varepsilon M(e) \qquad \forall\, e \varepsilon C^{\cdot}_{o} .$$

The coboundary $\delta^{\circ}\varphi$ of a O–cochain φ is given by the formula

$$\overline{\delta^{\circ}\varphi}\,(f) = f\bar{\varphi}\,(\alpha(f)) - \bar{\varphi}\,(\beta(f)) \qquad \forall\, f \varepsilon C ;$$

i.e. to give a l-coboundary is equivalent to give a principal crossed homomorphism of $C^{\cdot}$ to M.

Consequently, *the cohomology group of order* 1 *of* $C^{\cdot}$ *to* M *defined in* 4.1 *is the one constructed in the section* 2.

CONCLUSION

Although a notion of generalized module alread exists, module on a predditive category (i. e. on a ringoid), we have introduced the notion of module on a category. Same is not incompatible with the previous one. This two notions are connected as the notion of module on a group is with the notion of module on a ring. In the classical study of the cohomology of groups (see [6] p. 104), the modules on a group are defined as being the modules on the free ring of this group; we can also define the modules on a category $C^{\cdot}$ with a ringoid (see [1]) which is the free ring of $C^{\cdot}$ when $C^{\cdot}$ is a group. Regarding the generalized modules, refer to [10].

Consider the case in which the category $C^{\cdot}$ is a group whose the only unit is e. Let M be a $C^{\cdot}$ –module. The crossed product of M and $C^{\cdot}$, in the sense of 2.1, is the semi-direct product of M (e) and $C^{\cdot}$, in the sense of [6] p. 105, and H^1 (M) is the first cohomology group of $C^{\cdot}$ over M (e) defined in [6] p. 106. A group extension is a short exact sequence of groups $O \longrightarrow H^{+} \xrightarrow{\iota} K^{+} \xrightarrow{\pi} C^{\cdot} \longrightarrow O$, it is a sequence E: $H^{+} \xrightarrow{\iota} K^{+} \xrightarrow{\pi} C^{\cdot}$ such that E is an extension of categories in the sense of 4.1; the group $H^2(M)$ is the second cohomology group of $C^{\cdot}$ over M (e) defined in [6] p.

112. The canonical normalized Z -resolution of $C^{\cdot}$ is the Bar Resolution defined in [6] p. 114 and the cohomology groups H^k (M) are the classical cohomology groups of $C^{\cdot}$ with coefficient M (e) defined, for example, in [6] p. 115.

Consequently, the cohomology of small categories which we have constructed is a generalization of the classical cohomology of groups.

This cohomology is a particular case of the one considered in [2] where is defined the central cohomology of categories.

The homology defined in the chapter I is null in the case considered in the chapter II; we may find a homology of small categories in [11].

BIBLIOGRAPHY

[1] G. Hoff: "Sur une cohomologie des catégories" These, Paris 1970, not published.

[2] C. Ehresmann: "Cohomologie sur une catégorie" C.R. Acad. Sciences 259 p. 1683, Paris 1964.

[3] M. André: "Méthode simpliciale" Lecture Notes 32, Springer ed., Berlin 1967.

[4] C. Ehresmann: "Catégories et structures" Dunod ed., Paris 1964.

[5] B. Mitchell: "Theory of categories" Academic Press, New York 1965.

[6] S. Maclane: "Homology" Springer ed., Berlin 1963.

[7] S. Swierczkowski: "Triads in the homology of categories" (Report of the Midwest Category Seminar IV) Lecture Notes 137, Springer ed., Berlin 1970.

[8] H. Cartan, S. Eilenberg: "Homological algebra" Princeton Univ. Press, Princeton 1956.

[9] C. Ehresmann: "Produit croisé de catégories" C.R. Acad. Sciences 258 p. 2461, Paris 1964.

[10] G. Dartois: "Catégories préadditives et modules" These, Paris 1971.

[11] N. Baroudi: "Sur l'homolgie des catégories" These, Paris 1967.

Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi

Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara - 1975