COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématique

TOME 24 ANNÉE 1975

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by

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

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The Conditional Extended Linear Model

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(received Apnl, 1975)

ABSTRACT

In the conventional general linear model it is often possible to examine a more complete model that includes non-linear terms. The paper is concerned with conditional extended linear model and developes the procedure to test a general linear hypothesis involving some non-linear parameters.

1. INTRODUCTION

Milliken and Graybill [3] formulated the extended linear model and developed a procedure to test certain hypotheses about the vector of parameters occuring only in the nonlinear part of the model. Their techniques were restricted to test the null hypothesis, the model is linear against the alternative hypothesis that the model also involves nonlinear terms. Their model was not suitable to test hypotheses which involving fewer parameters of fewer linear functions of the parameters. The purpose of this paper is to define a conditional extended linear model which allows the flexability needed to test the additional hypotheses are of interest. When we test hypotheses of the form of Milliken and Graybill [3], the conditional extended linear model provides the same results as the extended linear model. The second section is a review to the results of the extended linear model, and the third section defines the conditional extended linear model and developes the procedure to test a general linear hypothesis involving nonlinear parameters.

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2. THE EXTENDED LINEAR MODEL

Before we define the extended linear model, we review the definition of the linear model.

2.1. *The Linear Model*

The linear model is expressed as

$$Y = X\beta + e \tag{2.1}$$

where Y is n x 1 vector of observed random variables distributed as MVN $(X\beta, \sigma^2 I)$, X is an n x p matrix of known constants of rank, q, $q < p$; β is a p x 1 vector of unknown parameters to be estimated, and e is an unobserved random vector $\sim$ MVN $(O, \sigma^2 I)$, where σ^2 is unknown. The extended linear model is essentially the linear model of (2.1) with a certain type of nonlinear part.

2.2. *Extended Linear Model.* (ELM)

The model with additional terms, called the extended linear model is expressed as

$$Y = X\beta + F\alpha + e \tag{2.2}$$

where F is an n x k matrix whose elements are known functions of the q unknown real variables in a basis set of $X\beta$, i. e., we shall write $F = \{f_{ij}(X\beta)\}$ and α is an unknown k x 1 vector of parameters. We assume that Rank$[X{:}F]=r$ almost everywhere with respect to Lebesque measure in the vector space spanned by the rows of $X\beta$ and we also assume that $q < r < n$.

2.3 *Testing Hypothesis in the ELM.*

We are interested in testing hypotheses about α for the model (2.2),but the elements of F are in general unknown. We first assume that the elements of F are known in order to describe which hypotheses are testable and obtain the corresponding sum of squares. Then we define hypotheses that are "conditionally testable", after we estimate the unknown elements of F.

LEMMA 2.1: For the linear model (2.1), the parametric functions $H\beta$ are estimable if and only if

$HX^{+}X = H$ or equivalently $r[X(I - H^{-}_{m}H)] = r(X) - r(H)$ where X is design matrix in the linear model, X^{+} is Moore-Penrose generalized inverse of X, H is an s x p matrix of arbitrary rank, $r[A]$ denotes the rank of the matrix A and H^{-}_{m} is a minimum norm generalized inverse of H (see [2, 4]).

DEFINITION 2.1: For the linear model (2.1), a hypothesis H_0: $H\beta = O$ vs. Ha: $H\beta \neq O$, is said to be testable if and only if the linear combinations $H\beta$ are estimable.

Consider testing the hypothesis $A\alpha = O$ for the ELM (2.2), where A is s x k matrix of known constants of rank $a < s$ selected such that the hypothesis is testable.

We see that the only linear combinations of α which are testable under the model in (2.2) when the elements of F are known are linear combinations of the rows of F which are orthogonal to the columns of X. With that in mind, partition the F matrix into two orthogonal parts, i. e., $F = XX^{-}F + (I - XX^{-})F$. Now (2.2) can be written

$$Y = X\beta + XX^{-}F\alpha + (I - XX^{-})F\alpha + e$$

or

$$Y = X\beta_1 + G\alpha + e \tag{2.3}$$

where $\beta_1 = \beta + X^{-}F\alpha$, $G = (I - XX^{-})F$; $X'G$ and $G'X$ equal to zero.

LEMMA 2.2: The hypothesis $A\alpha = O$ is testable under model (2.2) when the elements of F are known if and only if the hypothesis $A\alpha$ is testable under model (2.3) when the elements of G are known.

We can write model (2.1) as

$$Y = X\beta_1 + GA^{-}A\alpha + G(I - A^{-}A)\alpha + e$$

or

$$Y = X\beta_1 + \theta_1\alpha_1 + \theta_2\alpha_2 + e \tag{2.4}$$

where $\theta_1 = GA^{-}$, $\alpha_1 = A\alpha$, $\theta_2 = GP$, $\alpha_2 = P'\alpha$. If we factor the matrix $I - A^{-}A$ as $PP' = I - A^{-}A$, where $P'P = I$ (see Milliken [2]), we can write the model in (2.4) as

$$Y = X_0\beta_0 + e \tag{2.5}$$

where

$$X_0 = (X \,\vdots\, G) \begin{bmatrix} I & 0 & 0 \\ & & \\ 0 & A^- & P \end{bmatrix}, \; \beta_0 = \begin{bmatrix} I & X^-F \\ 0 & A \\ 0 & P' \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

The sum of squares due to error for the model (2.5) is

$$\begin{aligned} \text{SSERROR} &= Y'[I - X_0(X_0'X_0)^- X_0']Y \\ &= Y' (I - X_0X_0^-) Y, \end{aligned} \qquad (2.6)$$

where

$$X_0(X'_0X_0)^-X_0' = [X \;\; \theta_1 \;\; \theta_2] \begin{bmatrix} X'X & 0 & 0 \\ 0 & \theta'_1\theta_1 & \theta'_1\theta_2 \\ 0 & \theta'_2\theta_1 & \theta'_2\theta_2 \end{bmatrix}^- \begin{bmatrix} X' \\ \theta'_1 \\ \theta'_2 \end{bmatrix},$$

which is obtained because of the facts $X'\theta_1 = 0$ and $X'\theta_2 = 0$ and $X_0X_0^- = XX^- + URU'$, where

$$R = \begin{bmatrix} \theta'_1\theta_1 & \theta'_1\theta_2 \\ \theta'_2\theta_1 & \theta'_2\theta_2 \end{bmatrix}^- \qquad \text{and } U = (\theta_1, \; \theta_2).$$

We can express URU′ as

$$URU' = \theta_1 (\theta_1'\theta_1)^- \theta_1' + P_1 (P_1'P_1)^- P_1'$$

and

$$P_1 = [I - \theta_1(\theta_1' \theta_1)^- \theta_1'] \, \theta_2.$$

Hence, the sum squares error in (2.4) becomes

$$SSE = Y'(I - XX^- - \theta_1\theta_1^- - P_1P_1^-) \, Y. \qquad (2.7)$$

The next step is to derive the model restricted by the conditions of the null hypothesis. By subjecting the model to the conditions of the hypothesis $H_0 : \alpha_{\blacktriangle} = 0$, we get the restricted model,

$$Y = X\beta_1 + \theta_2\alpha_2 + e$$

or

$$\begin{aligned} Y &= (X \;\; \theta_2) \begin{bmatrix} \beta_1 \\ \alpha_2 \end{bmatrix} + e \\ &= X_{10}\beta_{10} + e. \end{aligned} \qquad (2.8)$$

The error sum of squares for the restricted model, called the sum of squares conditional error, is

$$SSE\ (COND) = SSE_0 = Y'(I - X_{10}X^-_{10})\ Y$$

where

$$X_{10}X^-_{10} = XX^- + (I - XX^-)\ \theta_2\ [(I - XX^-)\ \theta_2]^-$$
$$= XX^- + \theta_2\theta_2^-.$$

Then, we obtain

$$SSE_0 = Y'(I - XX^- - \theta_2\theta_2^-)\ Y. \quad (2.9)$$

The sum of squares due to the hypothesis obtained via the principle of conditional error is the difference between the error sum of squares in (2.9) and (2.6), i. e.,

$$SSH_0 = SSE_0 - SSE$$
$$= Y'(I - XX^- - \theta_2\theta_2^-)\ Y - Y'(I - XX^- - URU')\ Y$$
$$= Y'\ (\theta_1\theta_1^- + P_1P_1^- - \theta_2\theta_2^-)\ Y$$
$$= Y'\{\theta_1\theta_1^- + (I - \theta_1\theta_1^-)\theta_2\ [\ (I - \theta_1\theta_1^-)\ \theta_2]^- - \theta_2\theta_2^-\}Y$$
$$= Y'\ \{\ (I - \theta_2\theta_2^-)\ \theta_1\ [\ (I - \theta_2\theta_2^-)\ \theta_1\ ]^-\ \}\ Y.$$

The above results are summarized in the following analysis of variance table, where

$$D_{123} = D_{12} - D_{12}\theta_2\ (\theta'_2D_{12}\theta_2)^-\ \theta'_2D_{12}$$

$$SSE = Y'D_{123}Y$$

$$D_{12} = D_1 - D_1\theta_1\ (\theta_1'\ D_1\theta_1)^-\ \theta_1'\ D_1$$

$$D_1 = I - XX^-$$

$$F_1 = XX^-$$

$$F_2 = D_1\theta_1\ (\theta'_1D_1\theta_1)^-\ \theta_1D_1$$

$$F_3 = D_{12}\theta_2\ (\theta'_2D_{12}\theta_2)^-\ \theta'_2D_{12}$$

ANALYSIS OF VARIANCE

SOURCE	D. F.	SUM OF SQUARES
Estimation	$r\,(X'_0X_0)$	$Y'(XX^- + \theta_1\theta_1^- + P_1P_1^-)\,Y$
β_1	$r\,(X'X) = q$	$Y'(XX^-)\,Y$
$\alpha_1 \mid \beta_1$	$r\,(\theta'_1D_1\theta_1) = r\,(\theta'_1\theta_1) = r_0$	$Y'F_2Y = Y'(\theta_1\theta_1^-)\,Y$
$\alpha_2 \mid \alpha_1,\ \beta_1$	$r\,(\theta'_2D_{12}\theta_2) = r\,(P'_1P_1^-) = r_2$	$Y'F_3Y = Y'P_1P_1^-\,Y$
$\alpha_1,\ \alpha_2 \mid \beta_1$	$r\,(\theta'_1D_1\theta_1 + \theta'_2D_{12}\theta_2) = r\,(URU') = r_3$	$Y'\,(F_2 + F_3)\,Y = Y'(\theta_1\theta_1^- + P_1P_1^-)\,Y$
Due to H_0	$r\,(URU' - \theta_2\theta_2^-) = r_3 - r_1$	$Y'\,\{(I - \theta_2\theta_2^-)\,\theta_1\,[\,(I - \theta_2\theta_2^-)\,\theta_1]^-\}\,Y$
ERROR	$n - r\,(X'_0X_0) = n - \{r_0 + q + r_2\}$	$Y'(I - F_1 - F_2 - F_3)\,Y$

In the table $\alpha_1 | \beta_1$ represents the effect of α_1 adjusted for β_1, and

$$\alpha_1, \alpha_2 | \beta_1 = R(\alpha_1, \alpha_2 | \beta_1) = R(\alpha_1, \alpha_2, \beta_1) - R(\beta_1)$$

$$= Y'(XX^- + \theta_1\theta_1^- + P_1P_1^-) Y - Y'(XX^-) Y$$

$$= Y'(\theta_1\theta_1^- + P_1P_1^-) Y = \text{Due to } (\alpha_1 \text{ and } \alpha_2$$

$$\text{after } \beta_1),$$

(see Searle [4]).

3. THE CONDITIONAL EXTENDED LINEAR MODEL AND ASSOCIATED DISTRIBUTIONAL PROPERTIES.

The quantities in section 2.1 were derived by assuming the elements of F were known, when in reality they are known functions unknown parameters of β. Since those elements are unknown, we need to estimate them before continuing the analysis. Model (2.2) is essentially a nonlinear model and a nonlinear estimation technique could be utilized to estimate all the parameters of the model. But if we define a new model, we can obtain test statistics for various hypotheses whose exact distributions are known.

DEFINITION 3.1. The conditional extended linear model is

$$Y = X\beta + \hat{F}\alpha + e \tag{3.1}$$

where the elements of F have been estimated by replacing $X\beta$ by some estimate that is restricted to be a linear function of X^+y. The name conditional model comes from assuming $E(Y | F = \hat{F}) = X\beta + \hat{F}\alpha$. We now define which functions of α are estimable for this conditional model.

DEFINITION 3.2. The linear combinations $A\alpha$ are conditionally estimable for model (3.1) if and only if

$$A\,[\,(I - XX^-)\,\hat{F}\,]^-[\,(I - XX^-)\,\hat{F}\,] = A.$$

Using the sums of squares in section 2.3, we will replace the elements in the matrices F, θ_1, θ_2, β_1, and P_1 by the values from the conditional model to provide pseudo quadratic forms. The following theorems investigate the distributional properties of these pseudo quadratic forms.

THEOREM 3.1: For the conditions in section (2.3) under the less than full rank null hypothesis $A\alpha = 0$, the pseudo quadratic form

$$\hat{E}_1 = \frac{1}{\sigma^2} Y'(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) Y$$

is distributed as a non-central chi-squared random variable with r_3 degrees of freedom, where $r(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) = r_3$ with probability one.

Proof. The proof consists of showing that the conditions of theorem (3.1) Graybill and Milliken [1] hold for the above pseudo quadratic form. We have $K = XX^-$ and $L = I - XX^-$, thus $KL' = 0$, then

(1) $L'(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) L = \hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-$

(2) $\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-$ is idempotent as

$$(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) = \hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^- + \hat{\theta}_1\hat{\theta}_1^-\hat{P}_1\hat{P}_1^- + \hat{P}_1\hat{P}_1^-\hat{\theta}_1\hat{\theta}_1^-$$

where

$$\hat{\theta}_1\hat{\theta}_1^-\hat{P}_1\hat{P}_1^- = \hat{\theta}_1\hat{\theta}_1^-(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2[(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2]^- = 0$$

and

$$\hat{P}_1\hat{P}_1^-\hat{\theta}_1\hat{\theta}_1^- = \hat{P}_1[\hat{\theta}'_2(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2]^-\hat{\theta}_2'(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_1\hat{\theta}_1^- = 0.$$

(3) $\quad tr(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) = r(\hat{\theta}_1\hat{\theta}_1^-) = r_3.$

(4) $\quad$ Under H_0, $E(Y \mid F = \hat{F}) = \mu$, thus $\lambda = \frac{1}{2\sigma^2}\mu'(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\mu$ is the noncentrality parameter, which can be reduced as,

$$2\sigma^2\lambda = \mu'(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\mu$$

$$= (\hat{\beta}_1'X' + \alpha_2'\hat{\theta}_2')(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)(X\hat{\beta}_1 + \hat{\theta}_2\alpha_2)$$

$$= \alpha_2'\hat{\theta}_2'(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\hat{\theta}_2\alpha_2$$

$$= \alpha_2'\hat{\theta}_2'\hat{\theta}_1\hat{\theta}_1^-\hat{\theta}_2\alpha_2 + \alpha_2'\hat{\theta}_2'(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2[\hat{\theta}_2'(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2]^-\hat{\theta}'_2(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2\alpha_2$$

$$= \alpha_2'\hat{\theta}_2'\hat{\theta}_1\hat{\theta}_1^-\hat{\theta}_2\alpha_2 + \alpha_2'\hat{P}_1'\hat{P}_1(\hat{P}_1'\hat{P}_1)^-\hat{P}_1'\hat{P}_1\alpha_2$$

$$= \alpha_2'\hat{\theta}_2'\hat{\theta}_1\hat{\theta}_1^-\hat{\theta}_2\alpha_2 + \alpha_2'[\hat{\theta}_2'(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2]\alpha_2$$

$$= \alpha_2'\hat{\theta}_2\hat{\theta}_2^-\alpha_2$$

$$= \alpha'(I - A^-A)\hat{G}'\hat{G}(I - A^-A)\alpha.$$

Thus under the null hypothesis the estimated sum of squares due to $\alpha_1, \alpha_2 | \hat{\beta}_1$ is distributed as a noncentral chi-squared random variable.

Theorem 3.2. For the conditions in section (2.3) under the less than full rank null hypothesis $A\alpha = 0$ the pseudo quadratic form

$$\hat{E}_0 = \frac{1}{\sigma^2} Y'(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-) Y$$

is distributed as a central chi-square random variable with $n - (r_0 + q + r_2)$ degrees of freedom, where $r(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-) = n - (r_0 + q + r_2)$ with probability one.

Proof: For $K = XX^-$ and $L = I - XX^-$;

(1) $(I - XX^-)(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-)(I - XX^-) = (I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^- = M^*$

(2) $M^{*2} = M^*$ is idempotent as

$$M^*M^* = I - XX^- - (I - XX^-)(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) - (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)(I - XX^-) + \hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-.$$

Since $XX^-\hat{\theta}_1\hat{\theta}_1^- = XX^-(I - XX^-)\hat{F}A[(I - XX^-)\hat{F}A]^- = O$ and

$$\hat{\theta}_1\hat{\theta}_1^- \, XX^- = 0,$$

$$\begin{aligned} X^-\hat{P}_1 &= X^-(I - \hat{\theta}_1\hat{\theta}_1^-)\hat{\theta}_2 \,[\,(I - \hat{\theta}_1\hat{\theta}_1^-)\,\hat{\theta}_2]^- \\ &= X^-\hat{\theta}_2[\,(I - \hat{\theta}_1\hat{\theta}_1^-)\,\hat{\theta}_2]^- \\ &= X^-(I - XX^-)\,\hat{\hat{F}}P \;[\,(I - \hat{\theta}_1\hat{\theta}_1^-)\,\hat{\theta}_2]^- \\ &= 0. \end{aligned}$$

Then we obtain $M^{*2} = M^*$.

(3) Trace $(M^*) = n - (r_0 + q + r_2)$ a constant.

(4) Under H_0, the non - centrality parameter is

$$\lambda = \frac{1}{2\sigma^2}\,\mu'(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-)\,\mu$$

where $E\,(Y\,|F = \hat{\hat{F}}) = \mu$. The value of λ is

$$\begin{aligned} 2\sigma^2\lambda &= (X\hat{\hat{\beta}}_1 + \hat{\theta}_2\alpha_2)'\,(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-)\,(X\hat{\hat{\beta}}_1 + \\ &\quad + \hat{\theta}_2\alpha_2) \\ &= \hat{\beta}'_1\,X'M^*X\hat{\beta}_1 + \alpha_2'\,\hat{\theta}_2M^*\,\hat{\theta}_2\alpha_2 + \hat{\beta}_1'\,X'\,M^*\,\hat{\theta}_2\alpha_2 \\ &\quad + \alpha_2'\,\hat{\theta}_2'\,M^*\,X\,\hat{\beta}_1 \\ &= \alpha_2'\,\hat{\theta}_2'\,M^*\,\hat{\theta}_2\alpha_2 \\ &= \alpha_2'\,\hat{\theta}_2'\,(I - XX^-)\,\hat{\theta}_2\alpha_2 - \alpha_2'\,\hat{\theta}_2'\,(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\,\hat{\theta}_2\alpha_2 \\ &= \alpha_2'\,\hat{\theta}_2'\,\hat{\theta}_2\alpha_2 - \alpha_2'\,\hat{\theta}_2'\,XX^-\,\hat{\theta}_2\alpha_2 + \alpha_2'\,\hat{\theta}_2'\,\hat{\theta}_2\alpha_2 \\ &= 0. \end{aligned}$$

Theorem 3.3. If the less than full rank hypothesis $H_0 : A\alpha = 0$ is conditionally testable then the sum of squares due to the hypothesis is given by

$$SSH_0 = Y'\,\{(I - \hat{\theta}_2\hat{\theta}_2^-)\,\hat{\theta}_1\,[I - \hat{\theta}_2\hat{\theta}_2^-)\,\hat{\theta}_1\,]^-\}\,Y$$

and $\frac{SSH_0}{\sigma^2}$ is distributed as a central chi-square random variable with $r_3 - r_1$ degrees of freedom and

$$r\{(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1[(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1]^-\} = r_3 - r_1$$

with probability one.

Proof: The proof consists of showing the four previously stated conditions.

(1) For $L = I - XX^-$ and $K = XX^-$

$$(I - XX^-)\{(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1[(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1]^-\}(I - XX^-)$$
$$= M^{**}$$

where

$$M^{**} = (I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1[(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1]^- = \hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^- - \hat{\theta}_2\hat{\theta}_2^-.$$

(2) $M^{**2} = M^{**}$ is idempotent.

Since $M^{**} = VV^-$, where $V = (I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1$, it can then be shown that

$$M^{**}\,M^{**} = VV^-\,VV^- = VV^- = M^{**}.$$

(3) $tr\,(M^{**}) = r\,(M^{**}) = r_3 - r_1$.

(4) Under H_0, the noncentrality parameter λ is

$$2\sigma^2\lambda = \mu' M^{**}\mu$$
$$= (X\hat{\beta}_1 + \hat{\theta}_2\alpha_2)'(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1[(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1]^-(X\hat{\beta}_1 + \hat{\theta}_2\alpha_2)$$
$$= 0.$$

Theorem 3.4. Under the hypothesis, H_0;

a) The ratio $F\,(H_0) = \dfrac{SSH_0/r_3 - r_1}{SSE/n\text{-}(r_0+q+r_2)}$ is distributed as a central F with $r_3 - r_1$ and $n - (r_0 + q + r_2)$ degrees of freedom.

b) The ratio $F = \dfrac{\hat{E}_1/r_3}{\hat{E}_0/n - (r_0+q+r_2)}$ is distributed as a non-central F with r_3 and $n - (r_0 + q + r_2)$ degrees of freedom.

Proof. To determine whether the two quadratic forms $Y'A_1Y$ and $Y'A_2Y$ are independently distributed we need to show the following nine conditions hold with probability one, (see [1]).

1. $L'A_1L = A_1$
2. $L'A_2L = A_2$
3. A_1 is idempotent.
4. A_2 is idempotent.
5. tr $(A_1) = m_1$, m_1 is constant positive integer.
6. tr $(A_2) = m_2$, m_2 is constant positive integer.
7. $\mu'A_1\mu = \lambda_1$ is constant.
8. $\mu'A_2\mu = \lambda_2$ is constant.
9. $A_1A_2 = 0$.

We have already shown the first 8 conditions, thus only condition 9 need be justified.

Let $N = \hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-$,

$M^* = I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-$,

and $M^{**} = \hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^- - \hat{\theta}_2\hat{\theta}_2^-$.

Now we have to show $NM^* = 0$ and $M^*M^{**} = 0$. For NM^*, we have

$$\begin{aligned} NM^* &= (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-) \\ &= (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)(I - XX^-) - (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)(\hat{\theta}_1\hat{\theta}_1^- \\ &\quad + \hat{P}_1\hat{P}_1^-) \end{aligned}$$

$$= -(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\, XX^-$$
$$= 0.$$

Next, we have

$$\begin{aligned} M^*M^{**} &= (I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-)(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^- - \hat{\theta}_2\hat{\theta}_2^-) \\ &= (I - XX^-)(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) - (I - XX^-)\,\hat{\theta}_2\hat{\theta}_2^- \\ &\quad - (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-) + (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\,\hat{\theta}_2\hat{\theta}_2^- \\ &= -(I - XX^-)\,\hat{\theta}_2\hat{\theta}_2^- + (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^-)\,\hat{\theta}_2\hat{\theta}_2^- \\ &= -\hat{\theta}_2\hat{\theta}_2^- + \hat{\theta}_2\hat{\theta}_2^- \\ &= 0. \end{aligned}$$

Theorem 3.5. For the conditions of section (2.2) consider testing the hypothesis H_0: $A\alpha = \alpha_0$ $(r(A) < r(\hat{G})$. The sum of squares due to this hypothesis is

$$\hat{E}_3 = (Y' - \alpha_0' A^- \hat{G}')\,(\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^- - \hat{\theta}_2\hat{\theta}_2^-)\,(Y - \hat{G}A^-\alpha_0)$$

and $\hat{E}_3/\sigma^2$ is distributed as a central chi-square random variable, under the null hypothesis.

Proof. The proof consists of deriving the model restricted by the null hypothesis. Write the model as

$$Y = X\beta + XX^-\hat{F}\alpha + (I - XX^-)\,\hat{\hat{F}}\alpha + e \tag{3.5.1}$$
$$= X\hat{\beta}_1 + \hat{G}A^-A\alpha + \hat{G}\,(I - A^-A)\alpha + e$$
$$Y - \hat{\theta}_1\alpha_0 = X\hat{\beta}_1 + \hat{\theta}_2\alpha_2 + e. \tag{3.5.2}$$

The sum of squares due to error for model (3.5.2) is

$$\hat{E}_{12} = (Y - \hat{\theta}_1\alpha_0)'\,(I - XX^- - \hat{\theta}_2\hat{\theta}_2^-)\,(Y - \hat{\theta}_1\alpha_0).$$

The sum of squares due to error from model (3.5.1) is

$$\begin{aligned} \hat{E}_{11} &= Y'(I - XX^- - \hat{\theta}_1\hat{\theta}_1^- - \hat{P}_1\hat{P}_1^-)\,Y \\ &= (Y - \hat{\theta}_1\alpha_0)'\,M^*\,(Y - \hat{\theta}_1\alpha_0) + Y'M^*\,\hat{\theta}_1\alpha_0 + \alpha_0'\,\hat{\theta}_1'\,M^*Y \\ &\quad + \alpha_0'\,\hat{\theta}_1'\,M^*\,\hat{\theta}_1\alpha_0 \end{aligned}$$

$$= (Y - \hat{\theta}_1\alpha_0)' M^*(Y - \hat{\theta}_1\alpha_0)$$

where $Y' M^* \hat{\theta}_1\alpha_0 = 0$, $\alpha_0' \hat{\theta}_1' M^* Y = 0$, and $\alpha_0' \hat{\theta}_1' M^* \hat{\theta}_1\alpha_0 = 0$. Thus, the sum of squares due to hypothesis is

$$\hat{E}_3 = \hat{E}_{12} - \hat{E}_{11} = (Y - \hat{\theta}_1\alpha_0)' (\hat{\theta}_1\hat{\theta}_1^- + \hat{P}_1\hat{P}_1^- - \hat{\theta}_2\hat{\theta}_2^-) (Y - \hat{\theta}_1\alpha_0).$$

Write $Z = Y - \hat{\theta}_1\alpha_0$, then under H_0, $E(Z) = X\hat{\beta}_1 + \hat{\theta}_2\alpha_2$. Noncentrality parameters is $\lambda = \frac{1}{2\sigma^2} E(Z)' M^{**} E(Z) = 0$.

$$\lambda = \frac{1}{2\sigma^2} (\hat{\beta}_1' X' + \alpha_2' \hat{\theta}_2') (I - \hat{\theta}_2\hat{\theta}_2^-) \hat{\theta}_1 [(I - \hat{\theta}_2\hat{\theta}_2^-)\hat{\theta}_1]^- (X\hat{\beta}_1 + \hat{\theta}_2\alpha_2)$$

$$= \frac{1}{2\sigma^2} \hat{\beta}_1 X' (I - \hat{\theta}_2\hat{\theta}_2^-) \hat{\theta}_1 [(I - \hat{\theta}_2\hat{\theta}_2^-) \hat{\theta}_1]^- X \hat{\beta}_1$$

$$= 0.$$

Thus the theorem is proved.

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ÖZET

Verilerin analizinde, genel lineer modeli lineer olmayan terimleri de kapsayan "Daha tam model" olarak düşünmek olanaklıdır. Bu nedenle, çalışmada genişletilmiş koşullu lineer model incelenmekte ve bazı lineer olmayan parametreleri içinde bulunduran genel lineer hipotezler için test yöntemi geliştirilmektedir.

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Ankara Üniversitesi Basımevi. Ankara–1976