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## Converse of the Maximum Modulus Theorem and Rings of Continuous Complex Functions


by

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# Converse of the Maximum Modulus Theorem and Rings of Continuous Complex Functions

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SUMMARY

It is shown among others that if $D$ is a region (open connected set) in the complex plane and $C(D)$ the algebra of continuous complex functions on $D$ with the property that for every member $f \in C(D)$ and every closed disc $\overline{W}$ in $D$, $\|f\|_W = \|f\|_{\partial W}$ then any two such regions are conformally equivalent. Moreover every $f$ is analytic.

**1. Converse of The Maximum Modulus Theorem.** It is known that [1]

**Theorem A.** Suppose $M$ is a vector space of continuous complex functions on the closed unit disc $\overline{U}$, with the following properties:

(a) $1 \in M$.

(b) If $f \in M$, then also $jf \in M$.

(c) If $f \in M$, then $\|f\|_U = \|f\|_{\partial U}$.

Then every $f \in M$ is analytic in $U$.

Here 1 is the function which assigns the number 1 to each $z \in U$, $j$ is the identity function $j(z) = z$, $\partial U$ is the boundary of $U$, the right and left side of the equality in (c) stand for *supremum.*

The following are immediate consequences of theorem A.

**Theorem 1.** Suppose $M$ is a *vector space* of continuous complex functions on the open unit disc $U$, with the following properties:

(a) $1 \in M$.

(b) If $f \in M$, then also $jf \in M$.

(c) If $f \in M$, then on every concentric closed disc $\overline{W}$ lying in $U$,

$$\|f\|_W = \|f\|_{\partial W}.$$

Then every $f \in M$ is analytic in $U$.

*Proof.* By theorem A every $f \in M$ is analytic in $W$, hence in $U$, since $W$ is arbitrary.

We have the following generalization.

**Theorem 2.** Suppose $M$ is a *vector space* of continuous complex functions on a region $D$, with the following properties:

(a) $1 \in M$.

(b) If $f \in M$, then also $jf \in M$.

(c) If $f \in M$, then on every closed disc $\overline{W}$ lying in $D$, $\|f\|_W = \|f\|_{\partial W}$.

Then every $f \in M$ is analytic in $D$.

*Proof.* Let $a \in D$ be any point, and $\overline{W} \subset D$ a closed disc, centre at $a$. By theorem A every $f \in M$ is analytic in $W$. Since $a$ is arbitrary it follows that every $f \in M$ is analytic in $D$. One can also argue as follows: Let $\bar{E}$ be any closed subregion in $D$. Then $\bar{E}$ can be covered by a finite number of discs $W$. Then by theorem A every $f \in M$ is analytic in $W$. So every $f \in M$ is analytic in $\bar{E}$.

**2. Generalization to Riemann surfaces.** Let $R$ be an abstract Riemann surface with respect to a given set of *local parameters* $\{\Psi_{p_0}(p)\}$, and a corresponding set of *parametric neighborhoods* $\{N_{p_0}\}$. For each $p_0 \in R$, $z = \Psi_{p_0}(p)$ is a homeomorphic mapping of $N_{p_0}$ onto the open unit disc $U$: $|z| < 1$ with $\Psi_{p_0}(p_0) = 0$. Moreover for any two overlapping parametric neighborhoods $N_{p_0}$, $N_{p_1}$ with $z' = \Psi_{p_1}(p)$ and $p_1 \in N_{p_0}$ then $z = \Psi_{p_0}(\Psi^{-1}_{p_1}(z'))$ is analyic at $z' = 0$.

D e f i n i t i o n. Given an abstract Riemann surface $R$, a function $F(p)$ defined on a region $D \subset R$ is said to be analytic in $D$ if

$$f(z) = F(\Psi^{-1}_{p_0}(z))$$

is analytic in $U$ for every $p_0 \in D$.

**Theorem 3.** Suppose $M$ is a vector space of continuous complex functions on the abstract Riemann surface $R$, with the following properties:

(a) $1 \in M$.

(b) If $F \in M$, then also $JF \in M$, where $J(\Psi_{p_0}^{-1}(z)) = j(z) = z$, for every $p_0 \in R$.

(c) If $F \in M$, then for every closed $\overline{N}_{p_0}$, $\|f\|_U = \|f\|_{\partial U}$.

Then every $F \in M$ is analytic in $R$.

*Proof.* Let $\bar{E}$ be any *compact* (*closed*) region in $R$. Then $\bar{E}$ can be covered by a finite number of $N_{p_0}$. Then by the foregoing definition and theorem A every $F \in M$ is analytic in $N_{p_0}$. Hence every $F \in M$ is analytic in $E$. By exhausting $R$ with regions $E$, it follows that every $F \in M$ is analytic in $R$.

**3. Rings of continuous complex Functions.** It is well known that [2, 3, 4] if $D$ and $D'$ are two *plane regions* such that there is an *isomorphism* between the rings of analytic functions on the two *regions*, then $D$ and $D'$ are conformally equivalent. This theorem has been extended to arbitrary Riemann surfaces [5].

We shall prove the same theorem by using a weaker form of analytic functions resting on a maximum modulus as developed in the preceding paragraphs. Using Florack's theorem as hypothesis,

**Theorem 4.** Let $C(U)$ be the ring of continuous complex functions on the open unit disc $U$ with the property that on every concentric closed disc $\overline{W} \subset U$, $\|f\|_W = \|f\|_{\partial W}$ whenever $f \in C(U)$. Let $\Phi$ be a ring automorphism of $C(U)$ onto itself such that $\Phi(c) = c$ for every complex constant $c$, then there exists a one-to-one conformal mapping $z' = \varphi(z)$ of $U$ onto itself such that

$$f'(z') = f \circ \varphi^{-1}(z')$$

for any $f \in C(U)$ with $f' = \Phi(f)$, Moreover every $f \in C(U)$ is analytic in $U$.

*Proof.* First of all, $C(U) \neq \varnothing$. For $j(z) = z$ belongs to $C(U)$, also $1 \in C(U)$, and hence all polynomials are in $C(U)$.

Following [3], for each $\alpha \in U$, let $I_\alpha$ be the set of all functions $f \in C(U)$ which satisfy $f(\alpha) = 0$. $I_\alpha$ is a principal ideal of

$C(U)$ generated by $f = z - a$. Furthermore, $I_a$ is a maximal ideal of $C(U)$. Indeed, for every $f \in C(U)$ there exists a complex number $f(a)$ such that $f(z) - f(a) \in I_a$. Hence $C(U)/I_a$ is isomorphic with the field of complex numbers, thereby implying that $I_a$ is maximal. Later we shall give an algebraic characterization of *maximal ideals* of type $I_a$.

Let $\Phi$ be a *fixed automorphism* of $C(U)$. It takes a function $f \in C(U)$ into a function $f' \in C(U)$, a set $S \subset C(U)$ into a set $S' \subset C(U)$. In particular, to every *maximal ideal* $I_a$ of $C(U)$ there corresponds under $\Phi$ a *maximal ideal* $I_a'$, of $C(U)$, and conversely.

Putting $a' = \varphi(a)$, then $z' = \varphi(z)$ yields a one-to-one mapping of $U$ onto itself.

To show that $z' = \varphi(z)$ is conformal, consider the identity mapping $j \in C(U)$. Now, $f_0 = \Phi^{-1}(j) \in C(U)$ is the function on $U$ that corresponds to $j$ under the *automorphism* $\Phi$. For any $a \in U$, $f_0(z) - f_0(a) \in I_a$, while $\Phi(f_0(z) - f_0(a)) \in I_a'$, i. e., $\Phi(f_0(z)) - \Phi(f_0(a)) = z - f_0(a) \in I_a' = I\varphi(a)$. Hence $a' = f_0(a) = \varphi(a)$. But $f_0 \in C(U)$, so $\varphi \in C(U)$. However, $C(U)$ is at the same time a vector space satisfying the conditions of theorem 1. Hence $\varphi$ is analytic in $U$. But since $\varphi$ is one-to-one then $\varphi$ is conformal on $U$. Moreover every $f \in C(U)$ is analytic in $U$.

More generally we can prove

**Theorem 5.** Let $D$ be a region in the complex plane, and $C(D)$ the ring of continuous complex functions on $D$ with the property that on every closed disc $W$ lying in $D$, $\|f\|_W = \|f\|_{\partial W}$, whenever $f \in C(D)$. Let $D, D'$ be two regions in the complex plane with the above property. Let $\Phi$ be a *ring isomorphism* of $C(D)$ onto $C(D')$ such that $\Phi(c) = c$ for every complex constant $c$, then there exists a one-to-one conformal mapping $z' = \varphi(z)$ of $D$ onto $D'$ such that

$$f'(z') = f o \varphi^{-1}(z'), \qquad z' \in D'$$

for any $f \in C(D)$ with $f' = \Phi(f), f' \in C(D')$. Moreover, every $f \in C(D)$ ($f' \in C(D')$) is analytic in $D$ ($D'$).

*Proof.* If $I_a$ is the maximal ideal of $C(D)$ generated by $f = z - a$, we see that, as $a$ runs through $D$, to every maximal ideal $I_a$ of $C(D)$

there corresponds under $\Phi$ a maximal ideal $I_a'$ of $C(D')$, and conversely. Putting $a' = \varphi(a)$, then $z' = \varphi(z)$ yields a one-to-one mapping of $D$ onto $D'$.

To show that $z' = \varphi(z)$ is conformal on $D$, consider the identity mapping $j \in C(D')$. Now, $f_0 = \Phi^{-1}(j) \in C(D)$ is the function on $D$ that corresponds to $j$ under the isomorphism $\Phi$. For any $a \in D$, $f_0(z) - f_0(a) \in I_a'$ while $\Phi(f_0(z) - f_0(a)) \in I_a'$, i. e., $\Phi(f_0(z)) - \Phi(f_0(a)) = z - f_0(a) \in I_a' = I_{\varphi(a)}$. Hence $a' = f_0(a) = \varphi(a)$. But $f_0 \in C(D)$, so $\varphi \in C(D)$. However, $C(D)$ is at the same time a *vector space* (in fact an *algebra*) satisfying the conditions of theorem 2. Hence $\varphi$ is analytic in $D$, and because of the one-to-one property it is conformal on $D$. Moreover, every $f \in C(D)$ is analytic in $D$.

We finally arrive at the *algebraic characterization* of the conformal equivalence of two Riemann surfaces.

**Theorem 6.** Let $R$ be an open Riemann surface and $C(R)$ the ring of continuous complex functions $F$ on $R$, with the property that on every closed $\overline{N_{p_0}}$, $\|f\|_U = \|f\|_{\partial U}$, whenever $F \in C(R)$. Let $R$, $R'$ be two open *Riemann surfaces* with the above property. Let $\Phi$ be a *ring isomorphism* of $C(R)$ onto $C(R')$ such that $\Phi(c) = c$ for every complex constant $c$, then there exists a one-to-one conformal mapping $\varphi \in C(R)$ of $R$ onto $R'$ such that

$$F'(p') = F o \varphi^{-1}(p')$$

for all $p' \in R'$ and all $F \in C(R)$ with $F' = \Phi(F)$, $F' \in C(R')$. Moreover, every $F \in C(R)$ ($F' \in C(R')$) is analytic in $R$ ($R'$).

***Proof.*** It depends on the fact that every $p \in R$ yields a *principal maximal ideal* of $C(R)$ [6]. Then, as $p$ runs through $R$, to *every maximal ideal* $I_p$ of $C(R)$ there corresponds under $\Phi$ a *maximal ideal* $I_p'$ of $C(R')$, $p' \in R'$, and conversely. Putting $p' = \varphi(p)$, the latter yields a one-to-one mapping of $R$ onto $R'$, such that $F'(p') = F o \varphi^{-1}(p')$.

Put $J(p) = j(z) = z$, Where $p = \Psi_{p_0}^{-1}(z)$. Assume $J$ is the identity function on $R'$. Then $F_0 = \Phi^{-1}(J) \in C(R)$ is the function on $R$ that corresponds to $J$ under $\Phi$. Hence for any $a \in R$, $F_0(p) - F_0(a) \in I_a$. In particular, we may choose a parametric neighborhood $N_{p_0}$ containing $a$, with $p \in N_{p_0}$ and assume that $F_0(p) - F_0(a)$

$\in I_a$ holds in $N_{p_0}$, here $p_0 = p_0(a)$. Now, $\Phi(F_0(p) - F_0(a)) \in I_a'$, or $z - F_0(a) \in I_a' = I_{\varphi(a)}$. Hence $a' = F_0(a) = \varphi(a)$. But $F_0(a) \in C(R)$, $\forall\, a \in R$, and so $\varphi \in C(R)$. However, $C(R)$ is at the same time a vector space satisfying the conditions of theorem 3. Accordingly, $\varphi$ is analytic in $R$, and therefore conformal since it is one-to-one. Moreover, every $F \in C(R)$ is analytic in $R$.

**4. Algebraic Characterization of Maximal Ideals of Type $I_a$.** To simplify, we shall give a proof of the algebraic characterization of maximal ideals of type $I_a$ of the ring $A(R)$ of all analytic functions on an open Riemann surface $R$. Let $e, f, g, h, \ldots$ denote elements of $R$.

Let $f \in A(R)$, we call $N(f) = \{p \in R, f(p) = 0\}$ the *zero set of* the function $f$.

Definitions. We say that a function $f$ divides a function $g$ if and only if $N(f) \subset N(g)$.

If $f_1$ and $f_2$ have no common zeros, i. e., $N f_1) \cap N(f_2) = \varnothing$, then we say that the *greatest common divisor of* $f_1$ and $f_2$ is 1.

It is shown that if the greatest common divisor of $f_1$ and $f_2$ is 1, then there exist $e_1, e_2 \in A(R)$ such that

$$1 = e_1 f_1 + e_2 f_2.$$

Now, let $I = (f)$ be a *principal ideal* generated by $f$. It is known that $I$ is *maximal* if and only if $f$ has exactly one *zero*, i. e., $I = I_a$. On the other hand,

**Theorem 7.** Suppose that f has exactly one *zero*. Then $A(R)/I$ is a *field* isomorphic with the field of Complex numbers, and conversely.

*Proof.* $I$ defines an equivalence relation in $A(R)$ as follows: two elements $h, g \in A(R)$ are said to be *congruent modulo* $f$, written $h \equiv g$ (*modulo* $f$) if $N(f) \subset N(h-g)$, i. e., $h-g \in I$.

Let $F = A(R) / I$ be the set of *residue classes* so obtained. We introduce operations of *addition* and *multiplication* in a natural way. Namely, if $C_1, C_2 \in F$ and $f_1 \in C_1$, $f_2 \in C_2$ then $C_1 + C_2$ and $C_1 C_2$ are defined to be the residue classes of $f_1 + f_2$ and $f_1 f_2$ respectively. It is a matter of routine to show that this definition is independent of the selection of $f_1$ and $f_2$. In conclusion, $F$ forms a *commutative ring* with *identity element* $E$ consisting of the elements of $A(R)$ congruent to 1 modulo $f$. There remains to show that $F$ is actually a *field*. So let $C_1$ be any *non - zero residue class*, i.e., if $f_1 \in C_1$, then $f_1$ is not *divisible* by $f$. Consequently, there exist $e_1$, $e_2 \in A(R)$ such that

$$e_1 f_1 + e_2 f = 1.$$

Then $e_1 f_1 - 1 = - e_2 f$ is divisible by $f$ and accordingly,

$$e_1 f_1 \equiv 1 \textit{ modulo } f.$$

Hence, if $C_2$ is the residue class of $e_1$, we have $C_2 C_1 = E$. This shows that every non-zero element of $F$ has an *inverse* relative to the multiplication in $F$, so $F$ is a *field*. Since every $g \in A(R)$ *is congruent mod* $f$ to a complex number, it is clear that in the *canonical* mapping of $A(R)$ onto $A(R)/I$ the fied of complex numbers is mapped onto $A(R)/I$. But this homomorphism is actually an isomorphism since its *kernel* contains only the *zero*.

Conversely, if $A(R)/I$ is isomorphic with the field of complex numbers then $I$ is maximal. Since $I = (f)$ is proper, i.e., does not contain the constant 1, $f$ must vanish at some point $p \in R$. By Florak's [7] theorem there is a function $g \in A(R)$ which has a simple zero at $p$ and vanishes nowhere else. Hence $(f) \subset (g)$. By the maximality of $(f)$ we have $(g) = (f)$, or $g \in (f)$, and $g = hf$, Namely, f has exactly one simple zero.

*Remark.* Our results hold for spaces that are compact and locally euclidean, the ring isomorphism between the rings over any two such spaces turns these spaces into a Riemann surface.

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## ÖZET

Özellikle gösterilmiştir ki, D, kompleks düzlemde bir bölge, C (D), D üzerinde sürekli kompleks fonksiyonların cebri ise, diğer taraftan , C (D) ye ait herbir f elemanı ve D de herbir $\overline{W}$ kapalı daire için $\|f\|_{W} = \|f\|_{\partial W}$ şartı sağlanmış ise, bu taktirde, yukardaki hassaları haiz herhangi iki bölge konform eşdeğerdir. Üstelik her f analitiktir.

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