COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématiques

TOME 26 ANNÉE 1977

K-th Mean Function of Entire Functions Defined by Dirichlet Series

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Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences
de l'Université d'Ankara

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K-th Mean Function of Entire Functions Defined by Dirichlet Series

by

J.S. GUPTA and SHAKTI BALA

ABSTRACT

Let $f(s) = \sum_{n \varepsilon N} a_n e^{s\lambda_n}$ be an entire function defined by an everywhere convergent Dirichlet series whose exponents are subjected to the condition $\lim_{n \to \infty} \sup \frac{\log n}{\lambda_n} = D \varepsilon R_+ \cup \{0\}$ (R_+ is the set of positive reals). The notion of K-th mean function I_k of f was interduced by the first author in [2]. We generalize I_k, and define $I_{k,r}$, $r \varepsilon R$, as

$$I_{k,r}(\sigma,f) = \frac{1}{e^{r\sigma}} \int_0^\sigma I_k(x,f)\, e^{rx}\, dx, \ \forall \sigma \varepsilon R,$$ and study some propertes of I_k and $I_{k,r}$ in this paper. Beside establishing the convexity of I_k we have derived some formulas for Ritt order and lower order of f in terms of I_k and $I_{k,r}$ which are improvements and generalizations of known ones.

AMS subject classification number: Primary 30A64 Secondary 30A62. Key Words: Entire function, Dirichlet series, maximum modulus, maximum term, rank, K-th mean function, convex function, Ritt order, lower order.

1. Let E be the set of mappings f: C $\to$ C (C is the complex field) such that the image under f of an element $s \varepsilon C$ is $f(s) = \sum_{n \varepsilon N} a_n e^{s\lambda_n}$ with $\lim_{n \to +\infty} \sup \frac{\log n}{\lambda_n} = D \varepsilon R_+ \cup \{O\}$ (R_+ is the set of positive reals), and $\sigma^f_c = +\infty$ (σ^f_c is the absissa of convergence of the Dirichlet series defining f); N is the set of natural numbers 0, 1, 2, ..., $<\lambda_n \mid n \varepsilon N>$ is a strictly increasing unbounded sequence of nonnegative reals, $s = \sigma + it$, $\sigma, t \varepsilon R$ (R is the field of reals), and $<a_n \mid n \varepsilon N>$ is a sequence in C. Since the Dirichlet series defining f converges for each complex s, f is an entire funtion. ,Alos since $D \varepsilon R_+$, we have ([1)], p. 168), $\sigma_a{}^f = +\infty$

($\sigma_a{}^f$ is the absicissa of absolute convergence of the Dirichlet series defining f), and that f is bounded on each vertical line Re (s) $= \sigma_o$.

Let

(1.1) $M(\sigma, f) = \sup_{t \varepsilon R} \{ | f(\sigma + it) | \}, \forall \sigma < \sigma^f{}_c$, be the maximum modulus of an entire function f ε E on any vertical line Re (s) $= \sigma$,

(1.2) $\mu(\sigma, f) = \max_{n \varepsilon N} \{ |a_n| e^{\sigma \lambda_n} \}, \forall \sigma < \sigma^f{}_c$, be the maximum term, for Re (s) $= \sigma$, in the Dirichlet series defining f, and

(1.3) $\nu(\sigma, f) = \max_{n \varepsilon N} \{ n \mid \mu(\sigma, f) = |a_n| e^{\sigma \lambda_n} \}, \forall \sigma < \sigma^f{}_c$,

be the rank of the maximum term.

The first author introduced ([2], p. 520) the notion of k-th mean function I_k, k ε Z_+ (Z_+ is the set of positive integers), of an entire function f ε E and defined it as

(1.4) $$I_k(\sigma, f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^k \, dt, \ \forall \sigma < \sigma^f{}_c.$$

We define the generalized k-th mean function $I_{k,r}$, r ε R, of f as

(1.5) $$I_{k,r}(\sigma, f) = \frac{1}{e^{r\sigma}} \int_0^{\sigma} I_k(x, f) e^{rx} \, dx, \ \forall \sigma < \sigma^f{}_c, \pi$$ and study

a few results pertaining to the functions I_k and $I_{k,r}$ in this paper.

2. First we establist two lemmas that we need later.

Lemma 1. For every entire function f ε E, I_k is an increasing function and log I_k is a convex function of σ.

Proof. We adopt the method of Titchmarch ([3], p. 174) to prove the lemma. Let $\sigma_1, \sigma_2, \sigma_3 \; \varepsilon$ R be such that $0 < \sigma_1 < \sigma_2 < \sigma_3$. Also let $g: R \to C$ and $h: C \to C$ be two functions defined, respectively, as

$$g(t_2) = \frac{|f(\sigma_2 + it_2)|^k}{\log |f(\sigma_2 + it_2)|}, \ \forall t_2 \ \varepsilon \ R,$$

and

$$h(s) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \log |f(s+it_2)| \, g(t_2)\, dt_2, \ \forall \ s \in C.$$

It is clear from the definition of h that it is analytic in the half-plane $Re(s) \leq \sigma_3$ and that $|h|$ attains its supremum on the boundary $Re(s) = \sigma_3$, say at $s = \sigma_3 + it_3$. Hence

$$I_k(\sigma_2, f) = h(\sigma_2) \leq h(\sigma_3 + it_3) \leq I_k(\sigma_3, f),$$

which shows that I_k increases steadily with σ.

We now choose β so that $e^{\beta\sigma_1} I_k(\sigma_1, f) = e^{\beta\sigma_3} I_k(\sigma_3, f)$.

Then

$$e^{\beta\sigma_2} I_k(\sigma_2, f) = e^{\beta\sigma_2} h(\sigma_2) \leq \sup_{\sigma_1 \leq Re(s) \leq \sigma_3} |e^{\beta s} h(s)| \leq e^{\beta\sigma_1} h(\sigma_1) \leq e^{\beta\sigma_1} I_k(\sigma_1, f),$$

whence

$$e^{\beta\sigma_2} I_k(\sigma_2, f) \leq e^{\beta\sigma_1} I_k(\sigma_1, f).$$

This gives

$$\log\left(\frac{I_k(\sigma_2, f)}{I_k(\sigma_1, f)}\right) \leq \beta(\sigma_1 - \sigma_2). \tag{2.1}$$

Since, by definition $\beta = \dfrac{1}{\sigma_1 - \sigma_3} \log\left(\dfrac{I_k(\sigma_3, f)}{I_k(\sigma_1, f)}\right)$, it follows, from (2.1), that

$$\log\left(\frac{I_k(\sigma_1, f)}{I_k(\sigma_2, f)}\right) \leqq \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} \log\left(\frac{(I_k(\sigma_3, f)}{I_k(\sigma_1, f)}\right),$$

or

$$\log I_k(\sigma_2, f) \leqq \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1} \log I_k(\sigma_1, f)) + \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_1} \log I_k(\sigma_3, f),$$ which

proves the convexity of $\log I_k$.

Lemma 2. For every entire function $f \in E$, $e^{r\sigma} I_k(\sigma, f)$ is an increassing convex function of $e^{r\sigma} I_{k,r}(\sigma, f)$.

Proof. We have

$$\frac{d\,(e^{r\sigma} I_k(\sigma,f))}{d\,(e^{r\sigma} I_{k,r}(\sigma,r))} = r + \frac{d}{d\sigma}(\log I_k(\sigma,f)),$$

where the derivative exists almost everywhere on any interval $[0,\sigma]$, $\sigma < \sigma^f_c$, since I_k is an increasing continuous function of σ. The lemma is now obvious, since $\log I_k$ is an increasing convex function of σ.

Theorem 1. *For every entire function* $f \in E$ *of Ritt order* $\rho \in R^*_+ U\{0\}$ *and lower order* $\lambda \in R^*_+ U\{0\}$ (R^*_+ *is the set of extended positive reals*),

$$(2.2)\qquad \begin{matrix}\rho\\ \lambda\end{matrix} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_k(\sigma,f)}{\sigma} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_{k,r}(\sigma,f)}{\sigma},$$

where $\log_2 x = \log \log x$.

Proof. We know ([1], p. 170) that

$$a_n e^{\sigma\lambda_n} = \lim_{T\to+\infty} \frac{1}{T} \int_{t_0}^{T} e^{-it\lambda_n} f(\sigma+it)\,dt, \ \forall\, n \in N.$$

Therefore

$$|a_n|\, e^{\sigma\lambda_n} \le \lim_{T\to+\infty} \frac{2}{2T} \int_{-T}^{T} |f(\sigma+it)|\,dt$$

$$\le A_k \Big(\lim_{T\to+\infty} \frac{2}{2T} \int_{-T}^{T} |f(\sigma+it)|^k\,dt\Big)^{1/k},$$

where

$$A_k = \begin{cases} 1 & \text{if } k = 1 \\ 4\left(\dfrac{\Gamma(1/2 + k/2(k-1))}{\sqrt{\pi}\;\Gamma(1+k/2(k-1))}\right)^{1-1/k}, & \text{if } k > 1, \end{cases}$$

by Holder's inequality. Hence

$$(2.3)\qquad (\mu(\sigma,f))^k \le A^k_k\, 2I_k(\sigma,f).$$

Also, from (1.4),

$$(2.4)\qquad I_k(\sigma,f) \le (M(\sigma,f))^k.$$

From (2.3) and (2.4) it, therefore, follows that

$$(2.5)\quad \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 \mu(\sigma,f)}{\sigma} \leq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_k(\sigma.f)}{\sigma}$$

$$\leq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 M(\sigma, f)}{\sigma}.$$

But ([4], Theroems (2.7) and (2.8))

$$(2.6)\quad \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 M(\sigma, f)}{\sigma} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 \mu(\sigma, f)}{\sigma}$$

and, by definition,

$$\begin{matrix}\rho\\ \lambda\end{matrix} = \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 M(\sigma, f)}{\sigma}$$

The first equality in (2.2) thus follows from (2.5), (2.6) and (2.7).

In order to establish the second equality in (2.2) we get, from (1.5),

$$I_{k,r}(\sigma, f) \leq I_k(\sigma, f)\frac{1}{r}(1 - e^{-r\sigma}).$$

Therefore

$$(2.8)\quad \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_{k,r}(\sigma, f)}{\sigma} \leq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_k(\sigma, f)}{\sigma}.$$

And, for any $\varepsilon \in R_+$,

$$I_{k,r}(\sigma + \varepsilon, f) \geq \frac{1}{e^{r(\sigma+\varepsilon)}} \int_{\sigma}^{\sigma+\varepsilon} I_k(x, f)\, e^{rx}\, dx,$$

$$\geq I_k(\sigma, f)\frac{1}{r}(1 - e^{-\varepsilon r}).$$

Therefore

$$(2.9)\quad \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_{k,r}(\sigma, f)}{\sigma} \geq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_k(\sigma,f)}{\sigma}.$$

Combining (2.8) and (2.9) we get the desired result.

Remarks. (i) For $k = 2$, we get, from Theorem 1,

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log_2 I_2(\sigma, f)}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix};$$

a result which was proved, respectively, by Gupta ([2], Theorem 3) under the condition that $\rho \varepsilon R_+$, and by Kamthan ([5], Theorem 1) under the condition that $\log |a_n / a_{n+1}| / (\lambda_{n+1} - \lambda_n)$ forms a nondecreasing function of n for $n > n_o$. Since we do not assume any of these conditions in Theorem 1 it generalizes and improves upon their results.

(ii) We also get, from Theorem 1, for $k = 2$,

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log_2 I_{2,r}(\sigma, f)}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix}.$$

This result was also proved, respectively, by Kamthan ([5], Lemma 1) under tne condition that $\log |a_n / a_{n+}1| / (\lambda_{n+1} - \lambda_n)$ forms a nondecreasing function of n for $n > n_o$, and by Juneja ([6], Theorem 3) under the condition that $\rho \varepsilon R_+$. Obiously Theorem 1 generalizes and improves upon their results also.

(iii) Giving a very lengthy proof, Kamthan has proved ([7], Theorem F) the first equality in (2.2). But ours is an alternative and shortest possible proof of it.

(iv) Bajpai has also established ([8], Theorem 1) the result in (2.2) but for entire functions $f \varepsilon E$ of finite Ritt order. Clearly we have improved upon his result also.

Theorem 2. *For every entire function* $f \varepsilon E$ *of Ritt order* $\rho \varepsilon R^*_+ U \{0\}$ *and lower order* $\lambda \varepsilon R^*_+ U \{O\}$,

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log (I_k(\sigma,f) / I_{k,r}(\sigma,f))}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix}. \tag{2.10}$$

Proof. We have, from the definitions of I_k and $I_{k,r}$,

$$\frac{d}{d\sigma} (r\,\sigma + \log I_{k,r}(\sigma, f)) = \frac{I_k(\sigma, f)}{I_{k,r}(\sigma, f)}.$$

Therefore

$$r(\sigma - \sigma_0) + \log I_{k,r}(\sigma, f) - \log I_{k,r}(\sigma_0, f) = \int_{\sigma_0}^{\sigma} \frac{I_k(x,f)}{I_{k,r}(x,f)}\, dx,$$

or

$$\log I_{k,r}(\sigma,f) = \log I_{k,r}(\sigma_0,f) + \int_{\sigma_0}^{\sigma} m_{k,r}(x,f)\, dx, \tag{2.11}$$

where

$$m_{k,r}(x,f) = \frac{I_k(x,f)}{I_{k,r}(x,f)} - r, \tag{2.12}$$

increases with σ by virtue of Lemma 2. Thus, for $\sigma > \sigma_0$, (2.11) gives

$$\log I_{k,r}(\sigma,f) - \log I_{k,r}(\sigma_0,f) < (\sigma - \sigma_0)\, m_{k,r}(\sigma,f).$$

Therefore

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_2 I_{k,r}(\sigma,f)}{\sigma} \leq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{k,r}(\sigma,f)}{\sigma},$$

or, using Theorem 1,

$$\begin{matrix}\rho\\ \lambda\end{matrix} \leq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{k,r}(\sigma,f)}{\sigma}. \tag{2.13}$$

Again, from (2.11), we get, for any $h \in R_+$,

$$\log I_{k,r}(\sigma + h, f) - \log I_{k,r}(\sigma_0,f) \geq \int_{\sigma}^{\sigma+h} m_{k,r}(x,f)\, dx \geq h\, m_{k,r}(\sigma,f),$$

which gives

$$\begin{matrix}\rho\\ \lambda\end{matrix} \geq \lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{k,r}(\sigma,f)}{\sigma}. \tag{2.14}$$

Combining (2.13) and (2.14), we get

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{k,r}(\sigma,f)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}. \tag{2.15}$$

The theorem now follows from (2.12) and (2.15).

Remark. Since we do not assume $\log|a_n/a_{n+1}|/(\lambda_{n+1}-\lambda_n)$ forms a nondecreasing function of n for $n > n_0$ in Theorem 2 it also generalizes and improves upon Theorem 2 of [5] for $k = 2$.

Theorem 3. *For every entire function* $f \varepsilon E$ *of finite Ritt order,*

(2.16) $$\log I_{k,r}(\sigma, f) \sim \log I_k(\sigma, f), \text{ as } \sigma \to +\infty.$$

This follows as a simple deduction of Theorem 2.

Remark. The result in (2.16) has also been proved by Bajpai ([8], p. 32). But ours is a shorter and different approach to arrive at it.

Theorem 4. *For every entire function* $f \varepsilon E$ *of infinite Ritt order and* $\varepsilon \in R_+$, *if* $\lambda_{N(\sigma,f)} \sim \lambda_{N(\sigma+D+\varepsilon,f)}$ *as* $\sigma \to +\infty$, *then*

(2.17) $$\liminf_{\sigma\to+\infty} \frac{\log I_k(\sigma, f)}{\lambda_{N(\sigma,f)}} = 0.$$

Proof. We have, from (2.3) and (2.4),

(2.18) $$\frac{1}{2A^k_k}(\mu(\sigma,f))^k \leq I_k(\sigma,f) \leq (M(\sigma,f))^k.$$

But, for any $\varepsilon \in R_+$ and $\sigma > \sigma_0(\varepsilon,f)$, we have ([9], p. 68).

(2.19) $$M(\sigma,f) < \mu(\sigma + D + \varepsilon,f).$$

From (2.18) and (2.19), we get, for any $\varepsilon \in R_+$ and $\sigma > \sigma_0(\varepsilon,f)$,

$$\frac{1}{2A_k^k}(\mu(\sigma,f))^k \leq I_k(\sigma,f) < (\mu(\sigma+D+\varepsilon,f))^k.$$

Hence

(2.20) $$\liminf_{\sigma\to+\infty} \frac{k \log \mu(\sigma,f)}{\lambda_{N(\sigma,f)}} \leq \liminf_{\sigma\to+\infty} \frac{\log I_k(\sigma,f)}{\lambda_{N(\sigma,f)}}$$

$$\leq \liminf_{\sigma\to+\infty} \frac{k \log \mu(\sigma + D + \varepsilon,f)}{\lambda_{N(\sigma,f)}}$$

But, since $\rho = +\infty$, we have, from the following result ([10], p. 87),

$$\liminf_{\sigma\to+\infty} \frac{\log \mu(\sigma,f)}{\lambda_{N(\sigma,f)}} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \limsup_{\sigma\to+\infty} \frac{\log \mu(\sigma,f)}{\lambda_{N(\sigma,f)}}$$

that

$$(2.21) \qquad \liminf_{\sigma\to+\infty} \frac{\log \mu(\sigma,f)}{\lambda_{N(\sigma,f)}} = 0.$$

The theorem now follows from (2.20) and (2.21)

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Universitey of Jammu,

Jammu–180001, J & K, India.

Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.
Prix de ce numéro : 5 TL (pour la vente en Turquie).
Prière de s'adresser pour l'abonnement à : Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara – 1978