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by

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A Note on the Matrix Transformations of L_r And L_s Defined in an Incomplete Space

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SUMMARY

The purpose of this note is to characterize some of the matrix transformations between the sequence spaces L_r and L_s defined in an incomplete seminormed complex linear space X with the seminorm p and the zero θ, where

$$L_r = \left\{ (x_k) : \sum_{k=1}^{\infty} [p(x_k)]^r < \infty ,\ 1 < r < \infty \text{ and } x_k \in X \right\}$$

and

$$L_s = \left\{ (x_k) : \sum_{k=1}^{\infty} [p\ (x_k)]^s < \infty ,\ 0 < s \leq 1 \text{ and } x_k \in X \right\}.$$

We also include a diagramatic representation of the theorems at the end of the paper.

1. INTRODUCTION

In this paper we are going to characterize the matrices which transform some of the sequence spaces defined in an incomplete seminormed complex linear space (X, p) with the seminorm p and the zero θ into another. In fact this is a similar type of work to the one in our first paper on the same subject, [1]. Therefore, it will be useful to recall some of the basic notations, definitions and lemmas (without proof) given in [1].

2. NOTATIONS AND DEFINITIONS

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} (n, k = 1, 2, ...) and S be the set of all sequences defined in (X, p). Let V and W be two subsets of S. Then, it is said that the matrix A defines a matrix transformation from V into W if for every sequence $x = (x_k) \in V$ the sequence $Ax = (A_n(x)) \in W$ where $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$. And it is denoted by $A \in (V, W)$.

We define the sequence space L_r and L_s as follows:

$$L_r = \{ x = (x_k): \sum_{k=1}^{\infty} [p(x_k)]^r < \infty, 1 < r < \infty \text{ and } x_k \in X\}.$$

and

$$L_s = \{ x = (x_k): \sum_{k=1}^{\infty} [p(x_k)]^s < \infty, 0 < s \leq 1 \text{ and } x_k \in X\}.$$

It is clear that the special case L_s for $s = 1$, e. i., L_1 is the set of absolute convergent series.

l_r and l_s are the corresponding spaces to L_r and L_s, respectively, in the case of $X = C$, i.e., the set of complex numbers. Throughout the paper Φ will denote the space of finite sequences of non-zero coordinates. R will be the set of row-finite matrices, i.e., the matrices whose rows are in Φ. And N will be the set of natural numbers.

3. LEMMAS

LEMMA 1. If X is incomplete, then L_r and L_s are also incomplete, [1].

LEMMA 2. If the sequence $(\sum_{k=1}^{\infty} a_{nk} x_k)_{n \in N}$ converges for every $(x_k) \in V$, where V is a space which has the unit vector $e^{(k)} = (\theta, \theta, ..., \theta, u, \theta, ...)$ with $u \in X$ $(p(u) > 0)$ in k^{th} place and θ otherwise, e. g., L_r, L_s then

$$(a_{nk})_{n \in N} \in c \quad (\forall\ k)$$

where c is the set of all convergent sequences in C, [1].

LEMMA 3. Let each of the v and w be one of the sequence space l_r and l_s, and V and W be the corresponding sequence spaces

L_r and L_s respectively. Then if a norm condition $f(A) < \infty$ is necessary for $A \in (v, w)$, it is also necessary for $A \in (V, W)$, [1].

LEMMA 4. Let $o < t < \infty$. Then in (X, p), the necessary and sufficient condition for $\sum_{k=1}^{\infty} a_k x_k$ to be convergent whenever $\sum_{k=1}^{\infty} [p(x_k)]^s < \infty$ is that $a = (a_k) \in \Phi$, [1].

LEMMA 5. Let $W \subset L_t$ $(0 < t < \infty)$. Then, the dual space of W is Φ, i.e.,

$$(W)^+ = \Phi.$$

4. SOME OF THE MATRIX TRANSFORMATIONS BETWEEN THE SPACES L_r AND L_s.

In this section we are going to give some of the matrix transformations between the spaces L_r and L_s.

THEOREM 1. In (X, p), $A \in (L_1, L_r)$ if and only if

(1) $A \in R$, i.e., $a_{nk} = 0$ for $k > k_0(n)$,

(2) $M = \sup_k \sum_{n=1}^{\infty} |a_{nk}|^r < \infty$.

PROOF. Sufficency : Suppose that the conditions hold and let $(x_k) \in L_1$. Then, using the Minkowski's inequality we get

$$\left(\sum_{n=1}^{\infty} \left[p\left(\sum_{k=1}^{k_o(n)} a_{nk} x_k\right)\right]^r \right)^{1/r}$$

$$\leq \sum_{k=1}^{k_o(n)} \left(\sum_{n=1}^{\infty} [|a_{nk}| p(x_k)]^r \right)^{1/r}$$

$$\leq \sum_{k=1}^{\infty} p(x_k) \left[\sum_{n=1}^{\infty} |a_{nk}|^r \right]^{1/r}$$

$$\leq M^{1/r}. \bar{p}(\bar{x}) < \infty$$

whenever $(x_k) \in L_1$.

N e c e s s i t y : According to Lemma 4, $A \in R$ and by Lemma 3, (2) is necessary.

THEOREM 2. Let $0 < s \leq 1$ and $1 < r < \infty$. Then, in (X, p), $A \in (L_s, L_r)$ if and only if

(1) $A \in R$, i.e., $a_{nk} = 0$ for $k > k_0(n)$,

(2) $M = \sup_k \sum_{n=1}^{\infty} |a_{nk}|^r < \infty$.

PROOF. The sufficiencies of the conditions are the results of being that

$$(L_1, L_r) \subset (L_s, L_r)$$

and Lemma 4 and Lemma 3 give the necessity of the conditions, respectively.

THEOREM 3. Let $0 < s \leq 1$. Then, in (X, p), $A \in (L_s, L_1)$ if and only if

(1) $A \in R$, i.e., $a_{nk} = 0$ for $k > k_0(n)$,

(3) $M = \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty$.

PROOF. S u f f i c i e n c y : By condition (1), we write that

$$\sum_{k=1}^{k_o(n)} a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} x_k,$$

and since $L_s \subset L_1$, we get

$$\sum_{n=1}^{\infty} p\left(\sum_{k=1}^{k_o(n)} a_{nk} x_k\right) = \sum_{n=1}^{\infty} p\left(\sum_{k=1}^{\infty} a_{nk} x_k\right)$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{nk}| \, p(x_k)$$

$$= \sum_{k=1}^{\infty} p(x_k) \sum_{n=1}^{\infty} |a_{nk}|$$

$$\leq M. \sum_{k=1}^{\infty} p(x_k)$$

$$< \infty$$

whenever $(x_k) \in L_s$.

Necessity: By Lemma 4, $A \in R$ and, Lemma 3 gives the necessity of (3).

THEOREM 4. Let $0 < s \leq q \leq 1$. Then, in (X, p), $A \in (L_s, L_q)$ if and only if

(1) $A \in R$, i.e., $a_{nk} = 0$ for $k > k_0(n)$,

(4) $M = \sup_k \sum_{n=1}^{\infty} |a_{nk}|^q < \infty$.

PROOF. Sufficiency: Let $(x_k) \in L_s$. $s \leq q$ implies that $(x_k) \in L_q$. Therefore, we get

$$\sum_{n=1}^{\infty} [p(\sum_{k=1}^{k_o(n)} a_{nk}\, x_k)]^q = \sum_{n=1}^{\infty} [p(\sum_{k=1}^{\infty} a_{nk}\, x_k)]^q$$

$$\leq \sum_{n=1}^{\infty} [\sum_{k=1}^{\infty} |a_{nk}| p(x_k)]^q$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{nk}|^q \, [p(x_k)]^q$$

$$= \sum_{k=1}^{\infty} [p(x_k)]^q \sum_{n=1}^{\infty} |a_{nk}|^q$$

$$\leq M. \sum_{k=1}^{\infty} [p(x_k)]^q$$

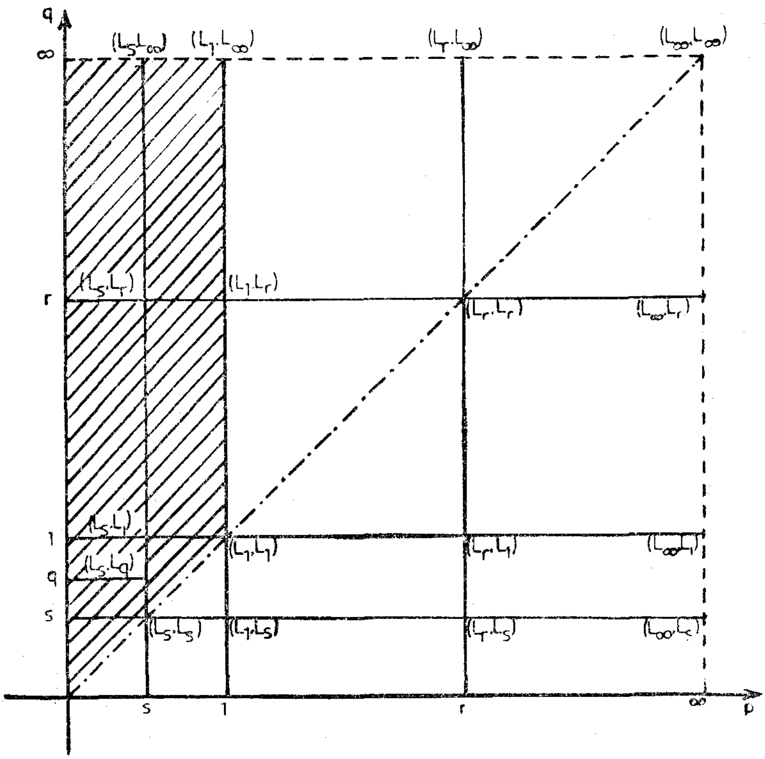
$$< \infty.$$

Necessity: Lemma 4 and Lemma 3 give the necessity of the conditions respectively.

The special case of the above theorem for $X = C$, i.e., (l_s, l_q) is due to Roles, [5].

DIAGRAMATIC REPRESENTATION

If we represent the matrix A which transform the space L_p into the space L_q by the point (p, q) in the Cartesien plane, as Mehdi did in [4], then we can give a geometrical explanation, in a sense, to the theorems we proved.



Generally, the purpose of this type of works is to characterize the points onto and outside of the square (!) with the corners (1, 1), (∞, 1), (∞, ∞), (1, ∞). Characterizations of the point in the shaded area were given in this paper. The points on the upper side of the square, i.e., the matrices (L_s, L_∞), (L_1, L_∞), (L_r, L_∞) and (L_∞, L_∞) had been given in [1] and [3] for the case X is incomplete. The points on the right and below side of the square will be investigated in our next paper.

But the points which are inside of the square are still unknown problems. Even the points on the diagnoal have not been solved yet, except the end points (L_1, L_1) and (L_∞, L_∞). The only paper on this which gives (l_2, l_2) for the case $X = C$, is due to L. Crone, [2]. The points below the square, i.e., the matrices (L_q, L_s), (L_1, L_s), (L_r, L_s) and (L_∞, L_s) are also the problems worthwhile to investigate.

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ÖZET

Bu araştırmamızda, tam olmayan yarı normlu bir kompleks lineer X uzayı üzerinde tanımlanmış L_r ve L_s dizi uzayları arasındaki matris dönüşümlerinden bazılarını karakterize ederek ispatladığımız teoremlerin geometrik bir yorumunu verdik.

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