COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématique

TOME 26 ANNÉE 1977

On the Zeros of Polynomials

by

V. K. JAIN

1

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences
de l'Université d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les disciplines scientifiques représentées à la Faculté.

La Revue, jusqu'a 1975 à l'exception des tomes I, II, III, était composée de trois séries:

Série A: Mathématique, Physique et Astronomie.

Série B: Chimie.

Série C: Sciences naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A_1: Mathématique
Série A_2: Physique
Série A_3: Astronomie
Série B : Chimie
Série C_1: Géologie
Série C_2: Botanique
Série C_3: Zoologie

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté. Elle accepte cependant, dans la mesure de la place disponible, les communications des auteurs étrangers. Les langues allemande, anglaise et française sont admises indifféremment. Les articles devront être accompagnés d'un bref sommaire en langue turque.

Adresse: Fen Fakültesi Tebliğler Dergisi Fen Fakültesi, Ankara, Turquie.

On the Zeros of Polynomials

V. K. JAIN

1. Govil and Rahman [1, Theorem 1] have proved the following theorem.

Theorem A. Let $p(z) = \sum_{k=0}^{n} a_k z^k \ (\neq 0)$ be a polynomial of degree n with complex coefficients such that for some $a > 0$

$$|a_n| \geqslant a|a_{n-1}| \geqslant a^2|a_{n-2}| \geqslant \ldots \geqslant a^{n-1}|a_1| \geqslant a^n|a_0|.$$

Then p (z) has all its zeros in $|z| \leq \left(\frac{1}{a}\right) K_1$, where K_1 is the greatest positive root of the trinomial equation

$$K^{n+1} - 2K^n + 1 = 0.$$

In the same paper [1], they also remark that Theorem A remains true if the polynomial has gaps and non-vanishing coefficients a_n, a_{n_1}, a_{n_2} satisfy

$$|a_n| \geqslant a^{n-n_1}|\ |a^{n_1}| \geqslant a^{n-n_2}|\ a^{n_2}| \geqslant \ldots.$$

We have sharpened the result for the polynomials having gaps and we prove

Theorem 1. Let $p(z) = a_n z^n + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \ldots + a_{n_k} z^{n_k}$, $a_n \neq 0$, be a polynomial of degree n with complex coefficients such that

(i) $n > n_1 > n_2 > \ldots > n_k$; $n_1, n_2, \ldots, n_k$ all being non-negative integers.

(ii) for some $a > 0$, the coefficients a_k's satisfy the condition

$$|a_n| \geqslant a^{n-n_1}|a_{n_1}| \geqslant a^{n-n_2}|a_{n_2}| \geqslant \ldots \geqslant a^{n-n_k}|a_{n_k}|. \qquad (1)$$

Then p (z) has all its zeros in $|z| \leq \left(\frac{1}{a}\right) K_2$, where K_2 is the greatest positive root of the equation

$$x^{n-n_k+1} - x^{n-n_k} - x^{n_1-n_k+1} + 1 = 0. \tag{2}$$

As remarked earlier, Theorem 1 is sharper than the bound given by Govil and Rahman [1]. For the sake of completeness, we shall verify this fact and for this, it is sufficient to prove that

$$K_2^n - K_2^{n-1} - K_2^{n-2} - \dots K_2 - 1 \leq 0. \tag{3}$$

For this we note that

$$1 \leq K_2 < 2. \tag{4}$$

Also K_2 will satisfy the equation

$$K_2^{n-n_k} - K_2^{n_1-n_k} - K_2^{n_1-n_k-1} - \dots - K_2 - 1 = 0. \tag{5}$$

Now we have

$$\begin{aligned} K_2^n - K_2^{n-1} - K_2^{n-2} - \dots - K_2 - 1 \\ &= K_2^n - K_2^{n-1} - K_2^{n-2} - \dots - K_2^{n_1-n_k+1} - (K_2^{n_1-n_k} \\ &\qquad + K_2^{n_1-k-1} + \dots + K_2^2 + K_2 + 1) \\ &= K_2^n - (K_2^{n-1} + K_2^{n-2} \dots + K_2^{n_1-n_k+1}) - K_2^{n-n_k}, \\ &\qquad \text{(by (5))} \\ &= K_2^n - K_2^{n_1-n_k+1}\left[\frac{K_2^{n-(n_1+1)+n_k} - 1}{K_2 - 1}\right] - K_2^{n-n_k} \\ &= \frac{K_2^{n-n_k}(K_2^{n_k} - 1)(K_2 - 1) - K_2^{n-n_k}(K_2^{n_k} - K_2^{n_1-n+1})}{K_2 - 1} \\ &\leq 0. \end{aligned}$$

Proof. If, $a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \dots + a^{n_k} z_{n_k} = p_1(z)$,

then for $|z| = R \left(> \frac{1}{a}\right)$, we have

$$|p_1(z)| \leqslant |a_{n_1}| \, |z|^{n_1} + |a_{n_2}| \, |z|^{n_2} + \dots + |a_{n_k}| \, |z|^{n_k}$$

$$= |a_{n_1}| R^{n_1} + |a_{n_2}| R^{n_2} + \dots + |a_{n_k}| R^{n_k}$$

$$= |a_{n_1}| R^{n_1} \left[1 + \left|\frac{a_{n_2}}{a_{n_1}}\right| \frac{1}{R^{n_1-n_2}} + \dots + \left|\frac{a_{n_k}}{a_{n_1}}\right| \frac{1}{R^{n_1-n_k}}\right]$$

$$\leq |a_{n_1}| R^{n_1} \left[1 + \frac{1}{(aR)^{n_1-n_2}} + \dots + \frac{1}{(aR)^{n-n_k}}\right], \text{(by (1)}$$

$$\leq |a_{n_1}| R^{n_1} \left[\sum_{v=0}^{n_1-n_k} \frac{1}{(aR)^v}\right]$$

$$= |a_{n_1}| R^{n_1} \frac{(aR)^{n_1-n_k+1} - 1}{(aR-1)\,(aR)^{n_1-n_k}} \tag{6}$$

Therefore, for every real θ, we have

$$|p(Re^{i\theta})| \geqslant |a_n| R^n - |p_1(Re^{i\theta})|$$

$$\geqslant |a_n| R^n - |a_{n_1}| R^{n_1} \frac{(aR)^{n_1-n_k+1} - 1}{(aR-1)(aR)^{n_1-n_k}}, \text{ (by (7))}$$

$$> 0$$

if

$$\frac{|a_n|}{a^{n-n_1}|a_{n_1}|} > \frac{R^{n_1}}{a^{n-n_1} R^n} \frac{(aR)^{n_1-n_k+1} - 1}{(aR-1)\,(aR)^{n_1-n_k}}$$

$$= \frac{(aR)^{n_1-n_k+1} - 1}{(aR)^{n-n_k}\,(aR-1)}$$

But $\dfrac{|a_n|}{a^{n-n_1}|a_{n_1}|} \geqslant 1$ by hypothesis, we conclude that $p(Re^{i\theta}) \neq 0$ if

$$(aR)^{n-n_k}\,(aR-1) > (aR)^{n_1-n_k+1} - 1.$$

On replacing (aR) by K, we get the result.

2. The following result is due to Cauchy ([2], p. 123).

Theorem B. All the zeros of $p(z) = a_0 + a_1 z + \ldots + a_n z^n$, $a_n \neq 0$, lie in the circle

$$|z| < 1 + \max | a_k / a_n |, \ k = 0,1,2, \ldots, n-1.$$

We sharpen this result in the case of polynomials having second and third leading coefficients equal to zero i. e. $a_{n-1} = 0$, $a_{n-2} = 0$. We prove

Theorem 2. Let $p(z) = \sum_{k=0}^{n} a_k z^k \quad (\neq 0)$ be a polynomial af degree n ($\geqslant 3$) such that $a_{n-1} = 0$, $a_{n-2} = 0$. Define

$$Q = \left(\max_{0 \leq j \leq n-3} \left| \frac{a_j}{a_n} \right| \right)^{\frac{1}{n}}.$$

Then all the zeros of such a polynomial lie in the circle

$$|z| < r \tag{8}$$

where r is the positive root of the trinomial equation

$$x^3 - x^2 - Q^n = 0. \tag{9}$$

Proof. Without loss of generality, we can assume that $Q \neq 0$. We have for $|z| > 1$,

$$|p(z)| \geqslant |a_n| \ |z|^n - |a_{n-3}| \ |z|^{n-3} - |a_{n-4}| \ |z|^{n-4} - \ldots - |a_0|$$

$$\geqslant |a_n| \ |z|^n - |a_n| \ Q^n \ |z|^{n-3} - |a_n| \ Q^n \ |z|^{n-4} - \ldots - |a_n| \ Q^n \ |z| - Q^n \ |a_n|$$

$$= |a_n| \ \left[|z|^n - Q^n \frac{|z|^{n-2} - 1}{|z| - 1} \right]$$

$$> |a_n| \ \left[|z|^n - Q^n \frac{|z|^{n-2}}{|z| - 1} \right]$$

$$= |a_n| \ |z|^{n-2} \ \left[\frac{|z|^3 - |z|^2 - Q^n}{|z| - 1} \right]. \tag{10}$$

Hence if $|z| \geqslant r$, then $|p(z)| > 0$. Therefore, the only zeros of $p(z)$ in $|z| > 1$ are those satisfy inequality (8). But, as all the zeros of $p(z)$ in $|z| \leq 1$ satisfy inequality (8) also, we have fully established Theorem 2.

REFERENCES

[1] **N. K. Govil and Q. I. Rahman,** *On the Eneström-Kakeya Theorem,* Tôhoku Math. Jour., Vol. 20 (1968), pp. 126–136.

[2] **M. Marden,** *Geometry of polynomials,* American Mathematical Society, Math. Surveys, No. 3, New York, 1966.

Department of Mathematics
Indian Institute of Technology
NewDelhi–110029
INDIA.

Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara – 1977