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A Note on the Degree of Approximation of Functions Belonging to the Lipschitz Class

by

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A Note on the Degree of Approximation of Functions Belonging to the Lipschitz Class

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ABSTRACT

Recently, the present author [1] has obtained the degree of approximation of $f \in \text{Lip}$ $(0 < \alpha \leq 1)$ by Borel's exponential mean of its Fouries. In this note we have shown that the degree of approximation obtained in [1] is best possible in certain sense.

1. INTRODUCTION. Let $\{s_n\}$ be a sequence of partial sums of the given series $\sum_{n=o}^{\infty} a_n$. Then Borel's exponential-mean of $\{s_n\}$ is defined by ([2], p. 182)

$$(1.1) \qquad e^{-p} \sum_{n=o}^{\infty} s_n \frac{p^n}{L^n} \quad (p>0).$$

Suppose f be a 2 π- periodic and L- integrable over $(-\pi, \pi)$. Then the Fourier series associated with f at a point x is given by

$$(1.2) \qquad \tfrac{1}{2}a_o + \sum_{n=e}^{\infty} (a_n \cos nx + b_n \sin nx).$$

A function $f \in \text{Lip}\ \alpha$ $(\alpha > 0)$ if

$$(1.3) \qquad f(x+h) - f(x) = O(|h|^{\alpha}).$$

We write

$$(1.4) \qquad s_m(x) = \tfrac{1}{2}a_o + \sum_{n=1}^{m} (a_n \cos nx + b_n \sin nx).$$

Throughout this note, K's denote positive constant not necessarily the same at each occurrence.

Recently, the present author [1] proved the following result:

THEOREM A. Let $f \in \mathrm{Lip}\,\alpha\ (0 < \alpha \leq 1)$ and let f be 2π-periodic and L-integrable over $(-\pi, \pi)$. Then the degree of approximation of f by Borel's exponential mean of its Fourier series is given by

$$(1.5)\qquad \max_{0\leq x\leq 2\pi} |f(x) - T_p(x)| = O\, p^{-\frac{\alpha}{2}}\ (p \to \infty),$$

where $T_p(x)$ is Borel's exponential mean of $(s_n(x)\}$

One of the aims of this note is to show that the degree of approximation of f by Borel's exponential mean of its Fourier series, as given by (1.5) is best possible. Precisely, we prove the following:

THEOREM. Let f be 2π- periodic and L- integrable over $(-\pi, \pi)$ and let $f \in \mathrm{Lip}\,\alpha\ (0 < \alpha \leq 1)$. Then there exists constant $K > 0$ such that

$$\max_{0\leq x\leq 2\pi} |f(x) - T_p(x)| > K\, p^{-\frac{\alpha}{2}}\ (p \geq P_o),$$

where $T_p(x)$ is Borel's exponential mean of $\{s_u(x)\}$.

2. PROOF OF THE THEOREM. Let.

$$\Phi_x(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

Then proceeding as in [1], we have

$$\pi\,|-T_p(x) + f(x)| = \int_o^{\pi} \frac{\Phi_x(t)}{\sin\frac{1}{2}t}\, e^{-p(1-\cos t)} \sin\{\tfrac{1}{2}t + p\sin t\}\, dt$$

$$= \left|\left(\int_o^{\frac{1}{p}} + \int_{\frac{1}{p}}^{\delta} + \int_{\delta}^{\pi}\right)\left(\frac{\Phi_x(t)}{\sin\frac{1}{2}t}\, e^{-p(1-\cos t)} \sin\{\tfrac{1}{2}t + p\sin t\}\, dt\right)\right|$$

$$= |J_1 + J_2 + J_3| \qquad \text{(suppose)}$$

$$\geq |J_2| - |J_1| - |J_3|.$$

However

$$\max_{0\leq x\leq 2\pi} |J_1| \leq \max_{0\leq x\leq 2\pi} \int_o^{\frac{1}{p}} \frac{|\Phi_x(t)|}{\sin\frac{1}{2}t}\, dt$$

$$\leq K_1/p^{\alpha}$$

and, as in I_2 of [1],

$$\max_{0\leq x\leq 2\pi} | J_3 | = O\left\{ \frac{1}{p} \int_{\delta}^{\pi} t^{\alpha-2}\{-\frac{\partial}{\partial t}\exp\{-2p\sin^2\frac{t}{2}\}\}\,dt \right\}$$

$$\leq K_3/p$$

Now

$$| J_2 | \geq | 2 \int_{\frac{1}{p}}^{\delta} \Phi_x(t)\, t^{-1}\, e^{-2p\sin^2 \frac{1}{2} t} \sin\{\tfrac{1}{2} t + p \sin t\}\, dt | -$$

$$- | \int_{\frac{1}{p}}^{\delta} \Phi_x(t) \{\operatorname{cosec} \tfrac{1}{2} t - 2/t\}\, e^{-2p\sin^2 \frac{1}{2} t} \sin\{\tfrac{1}{2} t + p \sin t\}\, dt |$$

$$= 2\, | J_{2,1} | - | J_{2,2} |, \text{ say,}$$

where

$$\max_{0\leq x\leq 2\pi} | J_{2,2} | = O\left\{ \int_{\frac{1}{p}}^{\delta} t^{1+\alpha}\, e^{-2p\sin^2 \frac{1}{2} t}\, dt \right\}$$

$$\leq K_{2,2}/p.$$

By the hypothesis, there exists a constant $K_4 > 0$ such that

$$K_4\, t^{\alpha} < \Phi_x(t) + 2\, K_4\, t^{\alpha}$$

therefore

$$| J_{2,1} | \geq | \int_{\frac{1}{p}}^{\delta} \frac{\Phi_x(t) + 2\, K_4\, t^{\alpha}}{t}\, e^{-2p\sin^2 \frac{1}{2} t} \sin\{\tfrac{1}{2}t + p \sin t\}\, dt |$$

$$- 2\, K_4\, | \int_{\frac{1}{p}}^{\delta} t^{\alpha-1}\, e^{-2p\sin^2 \frac{1}{2} t} \sin(\tfrac{1}{2} t + p \sin t)\, dt |$$

$$= | J_{2,1,1} | - 2\, K_4\, | J_{2,1,2} | \text{ say.}$$

Since $t^{\alpha-1}\, e^{-2p\sin^2 \frac{1}{2} t}$ is positive and non- increasing, it follows by the second mean value theorem that

$$| J_{2,1,2} | = p^{1-\alpha} e^{-2p\sin^2\frac{1}{2p}} \left| \int_{\frac{1}{p}}^{\delta_1} \sin(\tfrac{1}{2} t + p \sin t)\, dt \right| (p^{-1} \leq \delta^1 \leq \delta)$$

$$= O(p^{-\alpha}).$$

Hence

$$\max_{0 \leq x \leq 2\pi} 2 K_4 | J_{2,1,2} | < K_{2,1,2} p^{-\alpha}.$$

By the first mean value theorem,

$$| J_{2,1,1} | = | \sin(\tfrac{1}{2} t' + p \sin t') | \int_{\frac{1}{p}}^{\delta} \frac{\Phi_x(t) + 2 K_4 t^{\alpha}}{t} e^{-2p\sin^2 \frac{1}{2} t}\, dt \quad (p^{-1} \leq t' \leq \delta)$$

$$\geq | \sin(\tfrac{1}{2} t' + p \sin t') | \int_{\frac{1}{\sqrt{p}}}^{\delta} \frac{\Phi_x(t) + 2 K_4 t^{\alpha}}{t} e^{-2p\sin^2 \frac{1}{2} t}\, dt$$

$$> K_4 | \sin(\tfrac{1}{2} t' + p \sin t') | \int_{\frac{1}{\sqrt{p}}}^{\delta} t^{\alpha-1} e^{-2p\sin^2 \frac{1}{2} t}\, dt$$

$$\geq K_4 | \sin(\tfrac{1}{2} t' + p \sin t') | \int_{\frac{1}{\sqrt{p}}}^{\delta} t^{\alpha-1} e^{-\frac{1}{2}pt^2}\, dt$$

$$= K_4 | \sin(\tfrac{1}{2} t' + p \sin t') | \frac{1}{p} \int_{\frac{1}{\sqrt{p}}}^{\delta} t^{\alpha-2} \frac{d}{dt} (- e^{-\frac{1}{2}pt^2})\, dt$$

$$\geq K_{2,1,1} p^{-\frac{\alpha}{2}} \qquad (p > p'),$$

where the constant $K_{2,1,1}$ depends upon $\sin(\frac{1}{2} t' + p \sin t')$, α, δ and K_4. However the integral $J_{2,1,1}$ is not zero therefore the constant $K_{2,1,1}$ is positive.

Now, for $p > p'$ collecting the results we get

$$\max_{0 \leq x \leq 2\pi} \pi \mid f(x) - T_p(x) \mid$$

$$\geq K_{2,1,1}\, p^{-\frac{\alpha}{2}} - (K_1 + K_3 + K_{2,2} + K_{2,1,2})\, p^{-1}$$

$$= p^{-\frac{\alpha}{2}} \left[K_{2,1,1} - (K_1 + K_3 + K_{2,2} + K_{2,1,2})\, p^{-\frac{\alpha}{2}-1} \right]$$

And, for any given constant K' such that $K_{2,1,1} - K' > 0$, we can find a positive number $P_o = P_o(K') > p'$ such that

$$(K_1 + K_3 + K_{2,2} + K_{2,1,2})\, p^{\frac{\alpha}{2}-1} \leq K' \text{ for } p \geq po$$

and hence writing K for $\frac{1}{\pi}(K_{2,1,1} - K')$ we get

$$\max_{0 \leq x \leq 2\pi} \mid f(x) - Tp(x) \mid \geq K\, p^{-\frac{1}{2}\alpha},\ (p \geq P0).$$

This proves the theorem completely.

ÖZET

1 nolu çalışmasında yazar $f \in lip\ (0 < \alpha \leq 1)$ in yaklaşım derecesini f in Fourier serisinin Borel Üstel ortalaması ile elde etmiş olup, bu çalışma 1 de elde edilen yaklaşım derecesinin, bir anlamda, en iyi sonuç olduğunu göstermiştir.

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