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by

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A Theorem on the Integrability of Power Series

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ABSTRACT

The aim of this paper is to obtain a most general generalization of Heywood's theorem [1] where a_n is not ultimately positive.

1. P. Heywood [(1), Theorem 3 (i)] has proved that if C_n is ultimately positive, that $\sum\limits_{n=o}^{\infty} C_n = o$ and that $F(x) = \sum\limits_{n=o}^{\infty} C_n x^n$ where the radius of convergence of power series is unity for $o \leq x < 1$, then $(1-x)^{-1} F(x)$ is $L(0,1)$ iff $\sum C_n \log n$ converges.

The object of this note is to prove more general result of this type, in which we do not assume that C_n is ultimately positive.

2. *Theorem*: If $f(x) = \sum\limits_{n=o}^{\infty} C_n x^n$, where the series is convergent in $o \leq x < 1$. if $\sum\limits_{n=o}^{\infty} C_n = o$, and if $\sum\limits_{n=1}^{\infty} C_n \log n$ is convergent, then $\int\limits_{o}^{1-\delta} \frac{f(x)}{1-x} dx$ tends to a unique finite limit as $\delta \to +0$, moreover if the series $\sum\limits_{n=1}^{\infty} C_n \log n$ has bounded partial sums then $\int\limits_{o}^{1-\delta} \frac{f(x)}{1-x} dx$ is bounded as $\delta \to +0$.

Conversely, if $\int\limits_{o}^{1-\delta} \frac{f(x)}{1-x} dx$ tends to a anlque finite limit as $\delta \to +0$ and if $S_n \log n \to \alpha$, as $n \to \infty$, then $\sum\limits_{n=o}^{\infty} C_n \log n$ is

convergent where α is finite and $S_n = \sum\limits_{r=0}^{\infty} Cr$. Moreover

if $\int\limits_{0}^{1-\delta} \dfrac{f(x)}{1-x}\, dx$ is bounded as $\delta \to +\,o$, and if $S_n \log n$ is

bounded as $n \to \infty$, then the series has bounded partial sums.

Proof: We have

$$\int\limits_{0}^{1-\delta} \frac{f(x)}{1-x}\, dx = \sum_{n=o}^{\infty} \int\limits_{0}^{1-\delta} \frac{C_n\, x^n}{1-x}\, dx$$

$$= -\sum_{n=o}^{\infty} \int\limits_{0}^{1-\delta} \frac{C_n (1-x^n)}{(1-x)}\, dx$$

$$= -\sum_{n=1}^{\infty} \int\limits_{0}^{1-\delta} C_n (1 + x + \ldots + x^{n-1})\, dx$$

$$= -\sum_{n=1}^{\infty} C_n \left[(1-\delta) + \frac{(1-\delta)^2}{2} + \ldots + \frac{(1-\delta)^n}{n} \right]$$

Putting $1-\delta = e^{-\Lambda}$, we have

$$I_o = -\sum_{n=1}^{\infty} C_n \left[e^{-\Lambda} + \frac{e^{-2\Lambda}}{2} + \ldots + \frac{e^{-n\Lambda}}{n} \right]$$

Also, if $e^{-\Lambda} = 1 - \dfrac{1}{N^2}$, where N is an positive integer,

we have

$$\sum_{r=1}^{N+1} \frac{e^{-r\Lambda}}{r} = \sum_{r=1}^{N+1} \frac{1}{r} \left[1 - \frac{1}{N^2} \right]^r$$

$$= \sum_{r=1}^{N+1} \frac{1}{r} \left[1 - \frac{r}{N^2} + 0 \left(\frac{1}{N^2} \right) \right]$$

$$= \sum_{r=1}^{N+1} \frac{1}{r} + 0 \left(\frac{1}{N} \right)$$

$$= \text{Log } N + r^* + U_{N+1} + 0 \left(\frac{1}{N} \right)$$

where r^* is Euler's constant and u_n tends to zero as $n \to \infty$. It follows easily that, if N is fixed

$$\limsup_{M \to \infty} \frac{\sum\limits_{r=1}^{M} \dfrac{e^{-r\Lambda}}{r}}{\text{Log } M} = 0$$

Also, we have $\sum\limits_{n=N+2}^{\infty} \dfrac{e^{-n\Lambda}}{n \log n} = I_1 + I_2 + I$.

$$I_1 = \sum_{n=N+2}^{\overline{N}} \frac{e^{-n\Lambda}}{n \log n} \; , \; I_2 = \sum_{n=\overline{N}+1}^{N^2} \frac{e^{-n\Lambda}}{n \log n}$$

$$I_3 = \sum_{n=N^2+1}^{\infty} \frac{e^{-n\Lambda}}{n \log n} \; , \; \overline{N} = \left[\frac{N^2}{\text{Log Log } N} \right] ,$$

where $[x]$ denotes the greatest integer not exceeding x. It follows easily that $I_1 \to \log 2$, $I_2 \to 0$, $I_3 \to 0$, as $n \to \infty$. Now, we have

$$\sum_{n=1}^{\infty} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right] = \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right] +$$

$$+ \sum_{n=N+1}^{\infty} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= I_{1,1} + I_{1,2} \text{ say.}$$

$$I_{1,1} = \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{1}{r} \left[1 - \frac{1}{N^2} \right]^r \right]$$

$$= \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{1}{r} \left(1 - \frac{r}{N^2} + O\left(\frac{1}{N^2} \right) \right) \right]$$

$$= \sum_{n=1}^{N} C_n \left[\sum_{r=1}^{n} \frac{1}{r} \right] - \frac{1}{N^2} \sum_{n=1}^{N} n \, C_n + O\left(\frac{1}{N} \right)$$

$$= \sum_{n=1}^{N} C_n \, (\log n + r^* + u_n) + O\,(1)$$

$$= \sum_{n=1}^{N} C_n \log n + \sum_{n=1}^{N} C_n u_n - r^* C_o + O(1)$$

$$= \sum_{n=1}^{N} C_n \log n + H + O(1)$$

where $H = - r^* C_o + \sum_{n=1}^{\infty} C_n u_n$, r^* is Euler's constant and u_n

tends steadily to zero as $n \to \infty$ Also since by hypothesis, S_n Log $n \to \alpha$ as $n \to \infty$, we have

$$I_{2,2} = \sum_{n=N+1}^{\infty} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= \lim_{M \to \infty} \sum_{N+1}^{M} C_n \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= \lim_{M \to \infty} \sum_{N+1}^{M} (S_n - S_{n-1}) \left[\sum_{r=1}^{n} \frac{e^{-r\Lambda}}{r} \right]$$

$$= \lim_{M \to \infty} \left(S_M \sum_{r=1}^{M} \frac{e^{-r\Lambda}}{r} \right) - \lim_{M \to \infty} \sum_{N+2}^{M} S_{n-1} \frac{e^{-n\Lambda}}{n}$$

$$- S_N \sum_{r=1}^{N+1} \frac{e^{-\Lambda r}}{r}$$

$$= \lim_{M \to \infty} \left[(S_M \text{ Log } M) \frac{\sum_{r=1}^{M} \frac{e^{-\Lambda r}}{r}}{\text{Log } M} \right]$$

$$- \lim_{M \to \infty} \sum_{N+2}^{M} (S_{n-1} \text{ Log } n) \frac{e^{-n\Lambda}}{n \log n}$$

$$- (S_N \text{ Log } N) \frac{\sum_{r=1}^{N+1} \frac{e^{-r\Lambda}}{r}}{\text{Log } N}$$

and since we have proved that

$$\sum_{n+2}^{\infty} \frac{e^{-n\Lambda}}{n \log n} \longrightarrow \text{Log } 2, \text{ as } N \to \infty$$ then by (2.1) it fol-

lows that $\sum_{n=1}^{\infty} C_n \text{ Log } n$ converges or oscillates finitely, accor

ding as $\int_{0}^{1-\delta} \frac{f(x)}{1-x} dx$ converges or oscillates finitely as $\delta \to +0$

Since I_o is bounded as $\delta \to +0$, therefore $I_{2,2}$ is bounded because by hypothesis $S_n \text{ Log } n$ is bounded as $n \to \infty$. Since

$I_{1,1} + I_{1,2}$ it follows that $\sum_{n=1}^{\infty} C_n \log n$ has a bounded partial sum

because $\sum_{n=1}^{N} C_n \log n$ is bounded for all N.

This completes the proof of our theorem.

I am thankful those persons who helped me in preparing this paper.

ÖZET

Bu çalışmada, a_n bir yerden itibaren sürekli olarak pozitif olmamak üzere, [1] de verilen Heywood Teoreminin en geniş anlamda bir genelleştirilmesi elde edilmiştir.
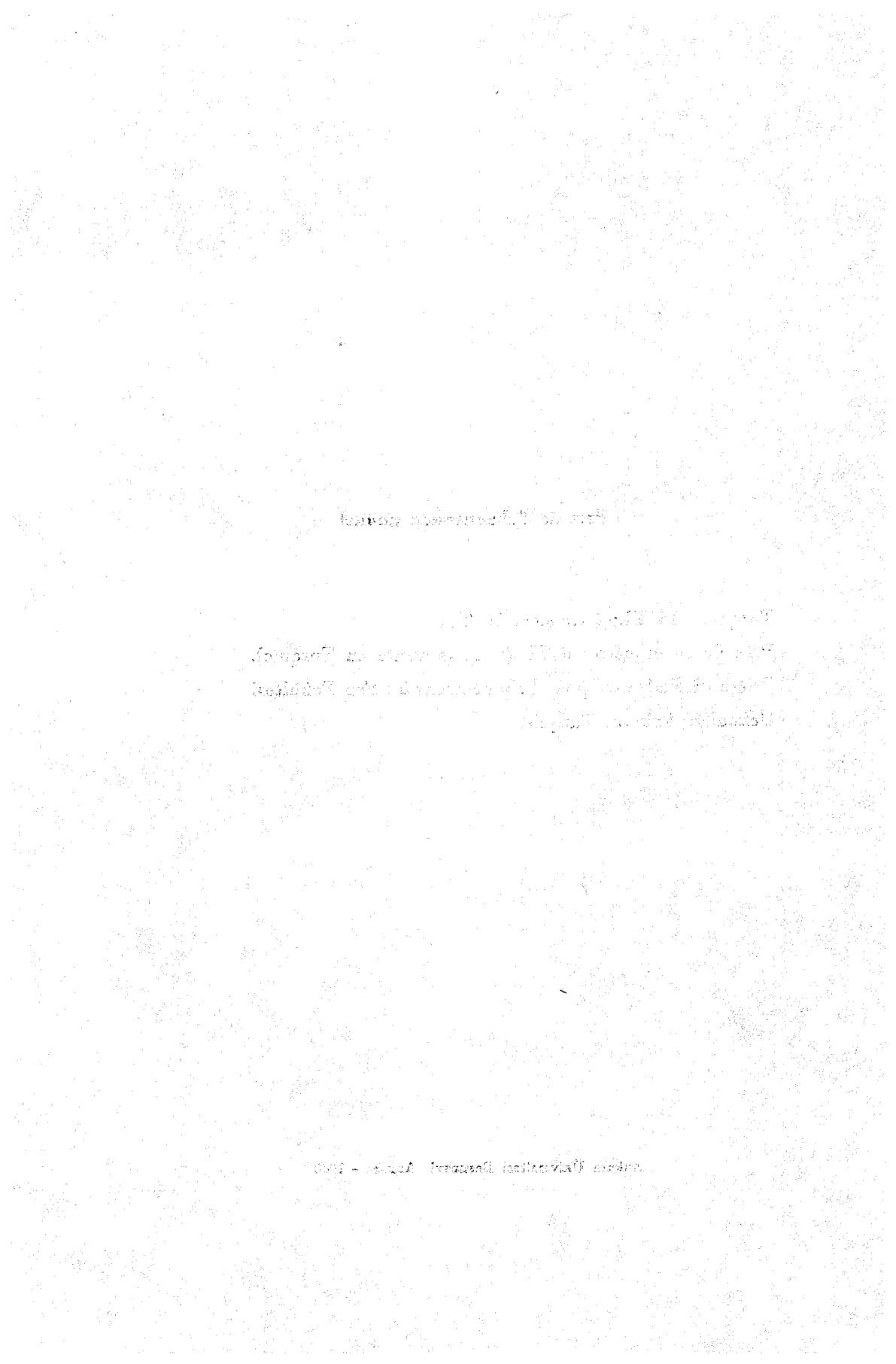
REFERENCES

1. P. Heywood, Integrability theorems for power series and Laplace transforms, Jour. London Math. Soc., 30, (1955), 302–310.

2. E.C. Titchmarch, Theory of Functions, Oxford, 1939.

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