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## A Class Of Ovoidal Laguerre Planes

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# A Class Of Ovoidal Laguerre Planes

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## ABSTRACT

We replace in the parabolic model of the classical Laguerre plane the parabolas  $y = ax^2 + bx + c$  by the polynomial curves  $y = ax^n + bx^m + c$ . Where  $n$  and  $m$  are even and odd integers, respectively. For each such a pair  $n, m$  with  $n > m > 0$  we obtain again a Laguerre plane  $\mathcal{L}_{mn}$  which is also ovoidal.

## 1. INTRODUCTION

A *Laguerre plane* is a system  $(\mathcal{P}, \mathcal{Z}, \varepsilon)$  which consists of a nonempty set  $\mathcal{P}$  of *points*, a nonempty set  $\mathcal{Z}$  of subsets (*cycles*) of  $\mathcal{P}$  and  $\varepsilon$  the set theoretical inclusion satisfying the following four axioms:

L1. For every three pairwise nonparallel points  $P, Q, R$  there exists a unique cycle  $z$  such that  $P, Q, R \in z$ .

(Two points are said to be *parallel* ( $//$ ) if and only if either  $P = Q$  or there is no cycle  $z \in \mathcal{Z}$  such that  $P \in z$  and  $Q \in z$ ).

L2. For each cycle  $z$  and each point  $P \notin z$ , there exists a unique point  $Q$  such that  $Q // P$  and  $Q \in z$ .

L3. For each cycle  $z$ , each point  $P \in z$ , and each point  $Q \in \mathcal{P} \setminus z$ ,  $P \not// Q$ , there exists a unique cycle  $z'$  such that  $P, Q \in z'$  and  $z \cap z' = \{P\}$ .

L4. For each  $P$ , there exist  $Q, R \in \mathcal{P}$  such that  $Q \neq P \neq R$ ,  $P // Q$  and  $P \not// R$ . Every cycle contains at least three points.

The origin of the Laguerre geometry is the geometry of oriented lines and oriented circles with nonnegative radius of the euclidean plane (see, Waerden-Smid [6] and Benz-Maurer [3]). In [1], Benz constructed the following class of Laguerre planes: Let  $F$  be an arbitrary field and  $V = F^3$  denote the three

dimensional vector space over  $F$ . Let  $O$  be an oval\*) in the plane  $\{(x,y,z) \in V: z = 0\}$ . Then  $(\mathcal{P}, \mathcal{Z}, \varepsilon)$  with  $\mathcal{P} = \{(x,y,z) \in V: (x,y,0) \in O\}$  and  $\mathcal{Z} = \{(x,y,z) \in \mathcal{P}: z = a x + b y + c\}: a,b,c \in F\}$  is a Laguerre plane in the above sense. Here  $\mathcal{P}$  is the set of points of an ovoidal cylinder and a cycle is intersection of  $\mathcal{P}$  with a plane which is not parallel to the axis of the cylinder. A Laguerre plane is *ovoidal* if it is isomorphic to a member of this class. (For a general definition, see Groh [4]). In [2], Benz shows that if  $O$  is given by  $x^2 + y^2 = 1$  then the corresponding Laguerre plane is isomorphic to the parabolic model  $(\mathcal{P}, \mathcal{Z}, \varepsilon)$  over  $\mathbb{R}$ , Where  $\mathcal{P} = \mathbb{R}^2 \cup \mathbb{R}$ ,

$$\mathcal{Z} = \{(x,y) \in \mathbb{R}^2: y = ax^2 + bx + c\} \cup \{a\}: a,b,c \in \mathbb{R}$$

and  $\mathbb{R}$  is the field of all real numbers. In order to obtain some new Laguerre planes, Hartman [5] replaced the parabolas in the above model by some particularly chosen curves.

The following are three basic properties of the Laguerre planes:

a) Parallelism is an equivalence relation on  $\mathcal{P}$ .

b) For every point  $P \in \mathcal{P}$  the *derived* incidence structure

$$A_P = (\mathcal{P} \setminus P, \{z: P \varepsilon z \varepsilon \mathcal{Z}\} \cup \{\bar{X}: X \in \mathcal{P}, X \parallel P\}, \varepsilon)$$

is an affine plane. (Where  $\bar{P}$  or  $\bar{X}$  denotes the equivalence class which contains  $P$  or  $X$ , respectively.)

c) In an ovoidal Laguerre plane all derived affine planes are desarguesian and isomorphic to each other.

The purpose of this paper is to give a class of ovoidal Laguerre planes by generalizing the parabolic model.

## 2. GENERALIZED PARABOLIC MODEL

Let  $\mathbb{R}$  denote the field of all real numbers. The central point of this work is

**Theorem 1.** Let  $n$  be an even integer and  $m$  an odd integer such that  $n > m > 0$ . Then the incidence structure

\* An oval  $O$  is a subset of a projective plane such that i) each line cuts  $O$  in at most two points, and ii) through each point  $P \in O$  goes exactly one tangent, i.e. a line intersecting  $O$  in exactly one point.

$\mathcal{L}_{nm} = (\mathcal{P}, \mathcal{Z}, \varepsilon)$  with  $\mathcal{P} = \mathbb{R}^2 \cup \mathbb{R}$  and  $\mathcal{Z} = \{(x, y) \in \mathbb{R}^2; y = a x^n + b x^m + c\} \cup \{a\}$ ;  $a, b, c \in \mathbb{R}$ , is a Laguerre plane.

*Proof.* It is obvious that  $\{a\} // \{\alpha\}$  for all  $a, \alpha \in \mathbb{R}$  and  $(x, y) // (u, v) \Leftrightarrow x = u$  for all  $x, y, u, v \in \mathbb{R}$ , and therefore  $\mathcal{L}_{nm}$  satisfies L2 and L4.

Two cases are possible for L1. In the case where given points are  $(x_i, y_i)$  with  $x_i \neq x_j, i \neq j, i, j = 1, 2, 3$ , let  $\det(x_i, y_i, z_i)$  denote the determinant function of which  $i$ th row vector is  $(x_i, y_i, z_i)$ . Let  $f(x_1, x_2, x_3) = \det(x_i^n, x_i^m, 1) = \delta$ . Considering any two of the variables as parameters, say  $x_2 = r$  and  $x_3 = s$ , and taking  $x_1 = x$  in  $f(x_1, x_2, x_3) = 0$  we obtain the trinomial equation

$$f(x, r, s) = \binom{r^m s^m}{x^n} + \lambda x^m + \mu = 0$$

where  $\lambda = (s^n - r^n)(r^m - s^m)^{-1}$  and  $\mu = (r^n s^m - r^m s^n)(r^m - s^m)^{-1}$ . By The Descartes rule of signs  $f(x, r, s)$  vanishes if and only if  $x = r$  or  $x = s$ . It follows that  $f(x_1, x_2, x_3) \neq 0$  if  $x_1 \neq x_2 \neq x_3 \neq x_1$ . Now, let  $[a, b, c]$  denote the cycle  $\{(x, y) \in \mathbb{R}^2; y = a x^n + b x^m + c\} \cup \{a\}$ . Using the equation of the cycle it is easily seen that  $a = \delta^{-1} \det(y_i, x_i^m, 1)$ ,  $b = \delta^{-1} \det(x_i^n, y_i, 1)$  and  $c = \delta^{-1} \det(x_i^n, x_i^m, y_i)$  for  $[a, b, c]$  containing the given points. In the case where given points are  $\{\alpha\}, (x_1, y_1), (x_2, y_2)$  with  $x_1 \neq x_2$ , it can be shown that the required cycle  $[a, b, c]$  is given by  $a = \alpha, b = (x_1^m - x_2^m)^{-1} \det(y_1 - \alpha x_1^n, 1)$  and

$$c = (x_1^m - x_2^m)^{-1} \det(x_1^m, y_1 - \alpha x_1^n).$$

For L3, let  $z = [a, b, c]$  and  $z' = [A, B, C]$ . For an euclidean point in  $z \cap z'$  the equation

$$F(x) = (A-a) x^n + (B-b) x^m + C-c = 0 \tag{1}$$

is valid. In the case where  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$  with  $x_1 \neq x_2$  necessarily  $A \neq a$ . From  $P, Q \in z'$  we have  $B = (x_1^m - x_2^m)^{-1} \det(y_1 - A x_1^n, 1)$ ,  $C = (x_1^m - x_2^m)^{-1} \det(x_1^m, y_1 - A x_1^n)$ . Additionally if  $x_1 \neq 0$  then  $C-c \neq 0$ , and by the Descartes rule of signs Eq. (1) has exactly two nonzero roots, one of which is  $x_1$ . These roots coincide if and only if  $F'(x_1) = 0$ . Combining this with Eq. (1) we obtain

$$A = k^{-1} [(mb + na x_1^{n-m})(x_1^m - x_2^m) - m(y_1 - y_2)], \text{ where } k = (n-m)x_1^{n-n}x_2^m x_1^{n-m} + m x_2^n, \text{ and clearly } x_1 \neq x_2 \text{ implies}$$

$k \neq 0$ . If  $x_1 = 0$  then  $C = c = y_1$ . Furthermore Eq. (1) has no real root other than 0 if and only if  $B = b$ , which implies that  $A = (y_2 - y_1 - b x_2^m) x_2^{-n}$ . In the case where  $P = (x_1, y_1)$  and  $Q = \{a\}$ , clearly  $A = a \neq a$  and  $C = y_1 - a x_1^n - B x_1^m$ . When  $x_1 \neq 0$ , a similar argument to the preceding case shows that the two real roots of Eq. (1) coincide if and only if  $F'(x_1) = 0$ , which gives  $B = b - n m^{-1} (a - a) x_1^{n-m}$ . When  $x_1 = 0$ , the nonexistence of a root of Eq. (1), distinct from 0, gives  $B = b$ . In the case where  $P = \{a\}$  and  $Q = (x_2, y_2)$  obviously  $A = a$ ,  $B = b$  and  $C = y_2 - a x_2^n - b x_2^m$  determine the required cycle uniquely.

*Corollary.* Let  $n$  be an even integer and  $m$  an odd integer such that  $n > m > 0$ . Then the incidence structure

$\mathcal{L}_{m/n} = (\mathcal{P}', \mathcal{Z}', \varepsilon)$  with  $\mathcal{P}' = \mathbb{R}^2 \cup \mathbb{R}$  and

$\mathcal{Z}' = \{(x, y) \in \mathbb{R}^2: y = a x^{n/m} + b x + c\} \cup \{a\}: a, b, c \in \mathbb{R}$  is a Laguerre plane isomorphic to  $\mathcal{L}_{nm}$ .

Proof follows from  $\varphi: (x, y) \rightarrow (x^m, y)$  being an isomorphism from  $\mathcal{L}_{nm}$  to  $\mathcal{L}_{n/m}$ .

By the above corollary, if  $n/m = 2$  then  $\mathcal{L}_{nm}$  is isomorphic to the parabolic model and consequently ovoidal. Furthermore it can be easily shown that the mapping  $(x, y) \rightarrow (x^{m^{-1}}, y + a x^{nm^{-1}})$  is an isomorphism between the real affine plane and the derived affine plane  $\mathcal{A}_P$  at  $P = \{a\}$ , for every  $a \in \mathbb{R}$ . In fact we have

**Theorem 2.** Every  $\mathcal{L}_{nm}$  is ovoidal.

*Proof.* Let  $O$  denote an oval given by

$\{(x, y, 0): x = \pm (1 + y)^{mn^{-1}} (1 - y)^{1-mn^{-1}}, -1 \leq y \leq 1\}$  in  $\mathbb{R}^3$ . Consider the ovoidal Laguerre plane  $(\mathcal{P}'', \mathcal{Z}'', \varepsilon)$  constructed with  $O$  by the Benz's method described in the introduction. For proof of the theorem it will be sufficient to show that this plane is isomorphic to  $\mathcal{L}_{n/m}$ . For this we need the stereographic projection

$\Psi: \mathcal{P}'' \setminus \{(0, 1, z): z \in \mathbb{R}\} \rightarrow xz\text{-plane}$

defined by  $\Psi((x, y, z)) = (x(1-y)^{-1}, 0, z(1-y)^{-1})$ . If

$\tilde{z} = \{(x, y, z): z = a x + b y + c, (x, y, 0) \in O\}$  then the image of  $\tilde{z} \setminus \{(0, 1, b+c)\}$  under  $\Psi$  is

$$\{(u,0,v): v = \frac{1}{2} (b+c) u^{n/m} + a u + \frac{1}{2} (c-b)\},$$

where  $u = x (1 - y)^{-1}$  and  $v = z (1 - y)^{-1}$ . Thus, if the definition of  $\Psi$  is extended by  $\Psi ((0,1,z)) = \{\frac{1}{2} z\}$ , for every  $z \in \mathcal{R}$ , it can be easily shown that  $\Psi$  gives an isomorphism from  $(\mathcal{P}'', \mathcal{Z}'', \varepsilon)$  to  $\mathcal{L}_{n/m}$ .

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#### ÖZET

Bu çalışmada, klasik Laguerre geometrisinin parabolik modelinin  $y = a x^2 + b x + c$  parabolleri,  $n$  çift ve  $m$  tek pozitif tam sayılar olmak üzere  $y = a x^n + b x^m + c$  polinom eğrileriyle değiştirilerek genelleştirilmekte ve bu özellikte her  $n, m$  ikilisi için bir Laguerre düzlemi elde edilmektedir. Son olarak da bulunan Laguerre düzlemlerinin hepsinin ovoidal olduğu gösterilmektedir.



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