

COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A₁: Mathématiques

TOME 28

ANNÉE 1979

On Strongly $F_{\mathcal{B}}$ -Summable Sequences

by

MURSALEEN

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

Comité de Rédaction de la Série A₁

B. Yurtsever, H. Hacısalihođlu, M. Oraç

Secrétaire de publication

A. Yalçın

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les disciplines scientifiques représentées à la Faculté.

La Revue, jusqu'à 1975 à l'exception des tomes I, II, III, était composée de trois séries:

Série A: Mathématiques, Physique et Astronomie.

Série B: Chimie.

Série C: Sciences naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A₁: Mathématiques

Série A₂: Physique

Série A₃: Astronomie

Série B: Chimie

Série C₁: Géologie

Série C₂: Botanique

Série C₃: Zoologie

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté. Elle accepte cependant, dans la mesure de la place disponible, les communications des auteurs étrangers. Les langues allemande, anglaise et française sont admises indifféremment. Les articles devront être accompagnés d'un bref sommaire en langue turque.

Adresse: Fen Fakültesi Tebliğler Dergisi, Fen Fakültesi, Ankara, Turquie.

On Strongly $F_{\mathcal{B}}$ - Summable Sequences

MURSALEEN

Aligarh Muslim University

Aligarh -202001 (INDIA)

(Received 20 October 1978, accepted 2 February 1979)

ABSTRACT:

In 1973, M. Steglitz defined the concept of $F_{\mathcal{B}}$ -convergence which is a generalization of almost convergence. Quite recently, Nanda introduced the spaces of strongly almost summable sequences. In the present paper author has generalized this concept of strongly almost summable sequences by defining the spaces of strongly $F_{\mathcal{B}}$ -summable sequences, and proves such spaces to be complete paranormed spaces under certain conditions, some topological result and characterization of strongly $F_{\mathcal{B}}$ -regular matrices have also been discussed.

1. Introduction.

Let S be the set of all sequences real or complex and l_{∞} denotes the Banach space of bounded sequences $x = \{x_k\}$ normed by $\|x\| = \sup_k |x_k|$. Lorentz [1] has introduced the concept of almost convergence by an application of Banach limits and characterized the space f of almost convergent sequences by means of the following property:

The sequence $x = \{x_k\}$ is almost convergent to the value f - $\lim x$, if

$$\lim_n \frac{1}{n+1} \sum_{k=i}^{i+n} x_k = f - \lim x \text{ (uniformly in } i = 0, 1, 2, \dots)$$

This criterion can also be formulated as: Let $\mathcal{B}_i = (\mathcal{B}_i^{(i)})$ be the sequences of matrices $\mathcal{B}_i^{(i)} = (b_{nk}^{(i)})$ with

$$b_{nk}^{(i)} = \begin{cases} \frac{1}{n+1} & i \leq k \leq i+n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus x is almost convergent to the each value f - $\lim x$, if

$$\lim_n \binom{(i)}{B_i x}_n = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} \binom{(i)}{b_{nk}(i)} x_k = f - \lim x$$

(uniformly, $i = 0, 1, 2,$)

Stieglitz [5] further generalized this concept by means of a given matrix sequence $\mathcal{B} = (B_i)$ with $B_i = (b_{nk}(i))$, $x \in l_\infty$ is $F_{\mathcal{B}}$ -convergent to the value $\text{Lim } \mathcal{B}x$, if

$$\lim_n (B_i x)_n = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} b_{nk}(i) x_k = \text{Lim } \mathcal{B}x$$

(uniformly, $i=0, 1, 2, \dots$).

The purpose of this paper is to introduce the spaces of strongly $F_{\mathcal{B}}$ -summable sequences.

Let $A = (a_{nk})$ be an infinite matrix of non-negative real numbers, and $p = \{p_k\}$ be a sequence such that $p_k > 0$. We write $Ax = \{A_n(x)\}$ if $A_n(x) = \sum_k a_{nk} |x_k|^{p_k}$ converges for each n .

We define (see Maddox [2])

$$[A, p]_0 = \{x: A_n(x) \rightarrow 0\};$$

$$[A, p] = \{x: A_n(x - le) \rightarrow 0 \text{ for some } l\};$$

and

$$[A, p]_\infty = \{x: \sup_n A_n(x) < \infty\}.$$

The spaces $[A, p]_0$, $[A, p]$ and $[A, p]_\infty$ are called the spaces of strongly summable to zero, strongly summable, and strongly bounded sequences, respectively.

Let

$$t_{n,i}(x) = \frac{1}{n+1} \sum_{j=i}^{i+n} A_j(x) = \sum_k a(i, k, n) |x_k|^{p_k}$$

where

$$a(i, k, n) = \frac{1}{n+1} \sum_{j=i}^{i+n} a_{jk}$$

We define (see Nanda [4])

$$[\hat{A}, p]_0 = \{x: t_{n,i}(x) \rightarrow 0 \text{ uniformly in } i\};$$

$$[\hat{A}, p] = \{x: t_{n,i}(x - l e) \rightarrow 0 \text{ for some } l \text{ (uniformly in } i)\};$$

and

$$[\hat{A}, p]_{\infty} = \{x: \sup_{n,i} t_{n,i}(x) < \infty\}.$$

The spaces $[\hat{A}, p]_0$, $[\hat{A}, p]$ and $[\hat{A}, p]_{\infty}$ are called the spaces of strongly almost summable to zero, strongly almost summable and strongly almost bounded sequences, respectively.

We now generalize these spaces by means of a given matrix sequence $\mathcal{B} = (B_i)$ with $B_i = (b_{nk}(i))$. We write

$$T_{n,i}(x) = \sum_k a(i,k,n) |x_k|^{p_k}$$

where

$$a(i,k,n) = \sum_j b_{nj}(i) a_{jk}$$

We now write

$$[Z^*, p]_0 = \{x: T_{n,i}(x) \rightarrow 0 \text{ uniformly in } i\};$$

$$[Z^*, p] = \{x: T_{n,i}(x - l e) \rightarrow 0 \text{ for some } l \text{ uniformly in } i\};$$

and

$$[Z^*, p]_{\infty} = \{x: \sup_{n,i} T_{n,i}(x) < \infty\}.$$

These are the spaces of strongly $F_{\mathcal{B}}$ -summable to zero, strongly $F_{\mathcal{B}}$ -summable, and strongly $F_{\mathcal{B}}$ -bounded sequences, respectively.

If x is strongly $F_{\mathcal{B}}$ -summable to l we write $x_k \rightarrow l [Z^*, p]$. A pair (A, p) will be called strongly $F_{\mathcal{B}}$ -regular if

$$x_k \rightarrow l \Rightarrow x_k \rightarrow l [Z^*, p].$$

These spaces of strongly $F_{\mathcal{B}}$ -summable sequences depend on the fixed chosen matrix $\mathcal{B} = (B_i)$. In case $\mathcal{B}_0 = (I)$ (unit matrix) these are equal to the spaces of strongly summable sequences and for $\mathcal{B}_i = \binom{(1)}{B_i}$ these are the spaces of strongly almost summable sequences.

2. Some important results

First we establish a number of lemmas.

Lemma 1. If $p \in l_\infty$, then $[Z^*, p]_0$, $[Z^*, p]$, and $[Z^*, p]_\infty$ are linear spaces over the complex field \mathbf{C} .

Proof. First we consider only $[Z^*, p]$. If $H = \sup p_k$ and

$K = \max(1, 2^{H-1})$, we have (see Maddox [3], p. 346)

$$(1) \quad |x_k + y_k|^{p_k} \leq K (|x_k|^{p_k} + |y_k|^{p_k})$$

and for $\lambda \in \mathbf{C}$

$$(2) \quad |\lambda|^{p_k} \leq \max(1, |\lambda|^H).$$

Suppose that $x_k \rightarrow l [Z^*, p]$, $y_k \rightarrow l' [Z^*, p]$ and $\lambda, \mu \in \mathbf{C}$.

Then we have

$$T_{n,i}(\lambda x + \mu y - (\lambda l + \mu l')e) \leq KK' T_{n,i}(x - le) + KK'' T_{n,i}(y - l'e)$$

where $K' = \sup |\lambda|^{p_k}$ and $K'' = \sup |\mu|^{p_k}$ and this implies that

$$\lambda x + \mu y \rightarrow (\lambda l + \mu l') [Z^*, p].$$

This terminates our proof.

Now $[Z^*, p]_0$ and $[Z^*, p]_\infty$ also can be treated similarly,

Lemma 2. $[Z^*, p] \subset [Z^*, p]_\infty$ if

$$(2.1) \quad \|A\| = \sup_n \sum_k \alpha(i, k, n) < \infty$$

Proof. Suppose that $x_k \rightarrow l [Z^*, p]$ and (2.1) holds. Now by the inequality (1)

$$T_{n,i}(x) = T_{n,i}(x - le + le)$$

$$(2.2) \quad T_{n,i}(x) \leq K T_{n,i}(x - le) + K \sum_k \alpha(i, k, n) |l|^{p_k}$$

$$\leq K T_{n,i}(x - le) + K (\sup |l|^{p_k}) \sum_k \alpha(i, k, n).$$

Therefore, $x \in [Z^*, p]_{\infty}$ and hence our proof is complete.

Lemma. 3 Let $p \in l_{\infty}$, then $[Z^*, p]_0$ and $([Z^*, p]_{\infty})$ (inf $p_k > 0$) are linear topological spaces paranormed by h defined by

$$h(x) = \sup_{n,i} |T_{n,i}(x)|^{1/M}$$

where $M = \max(1, \sup p_k)$. If (2.1) holds, then $[Z^*, p]$ has the same paranorm.

Proof. Clearly $h(0) = 0$ and $h(x) = h(-x)$. Since $M \geq 1$, by Minkowski's inequality it follows that h is subadditive. Now it follows from the inequality (2) that

$$h(\lambda x) \leq \sup |\lambda|^{P_{k/M}} h(x).$$

Therefore $x \rightarrow 0 \Rightarrow \lambda x \rightarrow 0$, λ fixed. Now let x be fixed and $\lambda \rightarrow 0$. Given $\varepsilon > 0$ there exists an integer N such that

$$(3.1) \quad T_{n,i}(\lambda x) < \varepsilon/2 \quad (\forall i, \forall n > N)$$

since $T_{n,i}(x)$ exists for all n , we write

$$T_{n,i}(x) = K(n), (1 \leq n \leq N)$$

and

$$\delta = \left(\frac{\varepsilon}{2 K(n)} \right)^{1/P_k}.$$

Then for $|\lambda| < \delta$,

$$(3.2) \quad T_{n,i}(\lambda x) < \frac{\varepsilon}{2} \quad (\forall i, 1 \leq n \leq N).$$

It follows from (3.1) and (3.2) that

$$\lambda \rightarrow 0 \Rightarrow \lambda x \rightarrow 0, x \text{ is fixed.}$$

This completes the proof for $[Z^*, p]_0$.

If $\inf p_k = \beta > 0$ and $0 < |\lambda| < 1$, then $\forall x \in ([Z^*, p]_{\infty})$

$$h^M(\lambda x) \leq |\lambda|^{\beta} h^M(x)$$

Therefore $[Z^*, p]_{\infty}$ has the paranorm h . If (2.1) holds it is clear from Lemma 2 that $h(x)$ exists for each $x \in [Z^*, p]$.

Hence the proof is complete.

Lemma 4. $[Z^*, p]_0$ and $[Z^*, p]_\infty$ are complete with respect to their paranorm topologies. $[Z^*, p]$ is complete if (2.1) holds and

$$(4.1) \quad \sum_k \alpha(i, k, n) \rightarrow 0 \text{ uniformly in } i.$$

Proof. Let $\{x^j\}$ be a Cauchy sequence in $[Z^*, p]_0$. Then there exists a sequence x such that $h(x^j - x) \rightarrow 0$ as $j \rightarrow \infty$. Since h is subadditive it follows that $x \in [Z^*, p]_0$. Therefore $[Z^*, p]_0$ is complete and similarly we can prove that $[Z^*, p]_\infty$ is complete.

We now consider $[Z^*, p]$. If (2.1) holds and $\{x^j\}$ be a Cauchy sequence in $[Z^*, p]$, Then there exists x such that $h(x^j - x) \rightarrow 0$. If (4.1) holds then from inequality (2.2) it is clear that $[Z^*, p] = [Z^*, p]_0$.

This terminates the proof.

Combining the above lemmas we obtain the following result.

Theorem 2.1. Let $p \in l_\infty$. Then $[Z^*, p]_0$ and $[Z^*, p]_\infty$ ($\inf p_k > 0$) are complete linear topological spaces paranormed by h . If (2.1) and (4.1) hold then $[Z^*, P]$ has the same property. Further if $p_k = p \forall k$, they are Banach spaces for $1 \leq p < \infty$ and p -normed spaces for $0 < p < 1$.

Theorem 2.2. Let $0 < p_k \leq 1$. Then $[Z^*, p]_0$ and $[Z^*, p]_\infty$ are locally bounded if $\inf p_k > 0$. If (2.1) holds, then $[Z^*, p]$ has the same property.

Proof. We consider only $[Z^*, p]_\infty$. Let $\inf p_k = \beta > 0$. If $x \in [Z^*, p]_\infty$ then there exists a constant $K > 0$ such that

$$\sum_k \alpha(i, k, n) |x_k|^{p_k} \leq K \quad (\forall n, i).$$

For this K and given $\delta > 0$ chose an integer $N > 1$ such that

$$N^\beta \geq \frac{K}{\delta}$$

Since $1/N < 1$ and $p_k \geq \beta$ we have

$$\frac{1}{N^{p_k}} \leq \frac{1}{N^\beta} \quad (\forall k)$$

Then for all n and i , we get

$$\begin{aligned} \sum_k a(i, k, n) \left| \frac{x_k}{N} \right|^{p_k} &\leq \frac{1}{N^\beta} \sum_k a(i, k, n) |x_k|^{p_k} \\ &\leq \frac{K}{N^\beta} \\ &\leq \delta \end{aligned}$$

Therefore by taking supremum over n and i we have

$$(2.2.1) \quad \{x : h(x) \leq K\} \subseteq N \{x : h(x) \leq \delta\}.$$

For every $\delta > 0$ there exists an integer $N > 1$ for which (2.2.1) holds and so

$$\{x : h(x) \leq K\}$$

is bounded. This completes the proof.

Theorem 2.3. Let $0 < p_k \leq 1$. Then $[Z^*, p]_0$ and $[Z^*, p]_\infty$ are r -convex for all r , $0 < r < \liminf p_k$. Moreover, if $p_k = p \leq 1$ $\forall k$, then they are p -convex.

$[Z^*, p]$ has the same property if condition (2.1) holds.

Proof. We shall prove our theorem only for $[Z^*, p]_\infty$. Let $x \in [Z^*, p]_\infty$ and $r \in (0, \liminf p_k)$. Then there exists k_0 such that $r \leq p_k$ ($\forall k > k_0$). Now define

$$f(x) = \sup_{n,i} \left[\sum_{k=1}^{k_0} a(i, k, n) |x_k|^r + \sum_{k=k_0+1}^{\infty} a(i, k, n) |x_k|^{p_k} \right].$$

Since $r \leq p_k \leq 1$ ($\forall k > k_0$), f is subadditive. Further for $0 < |\lambda| \leq 1$,

$$|\lambda|^{p_k} \leq |\lambda|^r \quad (\forall k > k_0)$$

Therefore for such λ we have

$$f(\lambda x) \leq |\lambda|^r f(x).$$

Now for $0 < \delta < 1$,

$$U = \{x : f(x) \leq \delta\}$$

is an absolutely r -convex set, for $|\lambda|^r + |\mu|^r \leq 1$ and $x, y \in U \Rightarrow$

$$\begin{aligned} f(\lambda x + \mu y) &\leq f(\lambda x) + f(\mu y) \\ &\leq |\lambda|^r f(x) + |\mu|^r f(y) \\ &\leq (|\lambda|^r + |\mu|^r) \delta \\ &\leq \delta \end{aligned}$$

If $p_k = p \forall k$, then for $0 < \delta < 1$,

$$\{x : f(x) \leq \delta\}$$

is an absolutely p -convex set. This can be obtained by a similar analysis. This completes the proof.

3. We now characterize the class of strongly $F_{\mathcal{B}}$ regular matrices.

Theorem 3.1 Let $0 < \theta \leq p_k \leq H < \infty$. Then (A, p) is strongly $F_{\mathcal{B}}$ -regular if and only if $A \in (C_o, F_o \mathcal{B})$.

Proof. Necessity: Suppose that (A, p) is strongly $F_{\mathcal{B}}$ -regular

Therefore

$$|x_k - l|^{1/p_k} \rightarrow 0 \Rightarrow \sum_k \alpha(i, k, n) |x_k - l| \rightarrow 0$$

uniformly in i . Again since $1/p_k \geq 1/H > 0$, Therefore by the following lemma (see Maddox [2], p. 347),

Lemma. If $p_k, q_k > 0$, then

$$c_o(q) c_o(p) \Leftrightarrow \liminf \frac{p_k}{q_k} > 0.$$

we have

$$x_k \rightarrow l \Rightarrow |x_k - l|^{1/p_k} \rightarrow 0.$$

Thus

$$x_k \rightarrow l \Rightarrow \sum_k \alpha(i, k, n) (x_k - l) \rightarrow 0$$

uniformly in i and hence $A \in (C_o, F_o \mathcal{B})$.

Sufficiency. Since $p_k \geq \theta > 0$, by above lemma

$$x_k \rightarrow l \Rightarrow |x_k - l|^{p_k} \rightarrow 0.$$

Again we have $A \in (c_0, F_{\mathcal{B}})$. Therefore $x_k \rightarrow l [Z^*, p]$.

This completes the proof.

4. Acknowledgement

The author would like to express his thanks to Dr. Z.U. Ahmad for suggestions and guidance.

ÖZET

1973 de M. Stieglitz, hemen hemen yakınsaklığın bir genelleştirilmesi olan $F_{\mathcal{B}}$ -yakınsaklık kavramını tanımlamıştır. Daha sonra Nanda, kuvvetli hemen hemen toplanabilir dizi uzaylarını tanımladı. Bu çalışmada yazar, kuvvetli $F_{\mathcal{B}}$ -toplanabilir dizi uzaylarını tanımlayarak kuvvetli hemen hemen toplanabilir dizi kavramını genelleştirmiştir ve bazı koşullar altında böyle uzayların paranormlu tam uzaylar olduğunu ispatlamış olup ayrıca bazı topolojik sonuçları elde etmiş ve kuvvetli $F_{\mathcal{B}}$ -regüler matrislerin karakterizasyonunu yapmıştır.

REFERENCES

- [1] Lorentz, G.G.: A Contribution to the theory of divergent sequences, Acta Math. 80 (1948), 167-190.
- [2] Maddox, I.J.: Spaces of strongly summable sequences, Quart. J. Math. Oxford ser. (2) 18 (1967).
- [3] Maddox, I.J.: Elements of functional Analysis (Camb. University Press (1970).
- [4] Nanda, S.: Some sequences spaces and almost convergence, J. Austral. Math. Soc. 22 (A) (1976), 446-455.
- [5] Stieglitz, M.: Eine Verrallgemeinerung des Begriffs des Fastknovergenz, Math. Japon 18 (1973), 53-70.

373 - MOHD. HABIB HALL
A. M. U. ALIGARH - 20200
INDIA

1900

...

1900

Prix de l'abonnement annuel

Turquie: 15 TL; Étranger: 30 TL.

Prix de ce numéro: 5 TL (pour la vente en Turquie).

**Prière de s'adresser pour l'abonnement à: Fen Fakültesi
Dekanlığı Ankara, Turquie.**

