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Absolute φ–Summability Factors

by

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Absolute φ-Summability Factors

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ABSTRACT: In this paper we prove the following theorem:

THEOREM: Let (λ_n) be a convex sequence such that $\Sigma \frac{\lambda_n}{n}$ is convengent. If there exist a $\mu > 0$ such that sequence $(n^{\mu-p} |\varphi_n|^p)$ is non-increasing and Σx_k is $[B, \log n, \varphi, 1]p$-bounded then $\Sigma \lambda_k x_k$ is $\varphi |C, 1|_p$ summable.

This theorem contains the theorem due to Mishra [3] as a special case for $\varphi_n = n^{1-p^{-1}}$.

1. **INTRODUCTION:** Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n, k = 1, 2, ...)$ and (φ_n) a sequence of complex numbers. Let Σx_k be a given infinite series with the sequence of partial sums (s_n). We denote the A transform of the sequence $s = (s_k)$ by $A_n(s)$ which is given by

$$A_n(s) = \sum_{k=1}^{\infty} a_{nk} s_k.$$

If

$$\sum_{n=1}^{\infty} | \varphi_n \bar{\triangle} A_n(s) |^p < \infty$$

for $p \geq 1$ then the series Σx_k is called $\varphi - |A|_p$ summable, where

$$\bar{\triangle} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $\varphi_n = n^{1-p^{-1}}$ or $\varphi_n = n^{\gamma+1-p^{-1}}$, then $\varphi - |A|_p$ summability is indentical with $|A|_p$ or $|A, \gamma|_p$ summability, respectively, [1], [2], [9].

The series Σx_k is called $[B, r, \varphi, \alpha]_p$ -bounded if

$$\sum_{v=1}^{n} \left| \frac{\varphi_v s_v}{v^\alpha} \right|^p = 0(r_n), \quad (n \to \infty),$$

where (r_n) is a non-decreasing sequence of positive real numbers and α is a real number.

In particular it is easy to show that if $r_n = \log n$, $\varphi_n = n^{1-p^{-1}}$ and $\alpha = 1$ then $[B, r, \varphi, 1]_p$ -boundedness is equivalent to $[R, \log n, 1]_p$ -boundedness.

For any sequence (λ_n) we use the following notation
$\triangle \lambda_n = \lambda_n - \lambda_{n+1}$, $\triangle^2 \lambda_n = \triangle (\triangle \lambda_n)$.

A sequence (λ_n) is said to be convex if $\triangle^2 \lambda_n \geq 0$ for every positive integer n. We require the following lemmas for the proof of our Theorem.

LEMMA 1: If (λ_n) is a convex sequence such that $\Sigma\, n^{-1} \lambda_n$ is convergent, then (λ_n) is non-negative and decreasing,

$n \triangle \lambda_n = o(1)$ and $\lambda_n \log n = o(1)$ as $n \to \infty$ ([4], [5]).

LEMMA 2: Let (λ_n) be a convex sequence such that the series $\Sigma\, n^{-1} \lambda_n$ is convergent, then

$$\sum_{n=1}^{m} \log(n+1) \triangle \lambda_n = 0(1), \quad (m \to \infty)$$

and

$$n \log(n+1) \triangle \lambda_n = 0(1) \text{ as } n \to \infty$$

([6], [7]).

LEMMA 3: Under the conditions of Lemma 2

$$\sum_{n=1}^{m} n \log(n+1) \triangle^2 \lambda_n = 0(1) \text{ as } m \to \infty$$

([6]).

LEMMA 4: Under the conditions of Lemma 2

$$\sum_{n=1}^{m} \log(n+1) \triangle (\lambda_n^p) = 0(1),\ p > 1 \text{ as } m \to \infty$$

([3]).

2. PROOF OF THE THEOREM

Let s_n^α and t_n^α be the n-th Cesaro means of order α $(\alpha > -1)$ of series Σa_n and of the sequence (na_n), respectively. Since $\bar{\triangle} s_n^\alpha = n^{-1} t_n^\alpha$, [8], it is enough to show that

$$\sum_{n=1}^{\infty} \left| \frac{\varphi_n T_n}{n} \right|^p < \infty \qquad (2.1)$$

where

$$T_n = (n+1)^{-1} \sum_{v=1}^{n} v \lambda_v x_v.$$

Now, applying Abel's transformation to the sum $\sum_{v=1}^{n} v \lambda_v x_v$,

we have

$$T_n = (n+1)^{-1} \sum_{v=1}^{n-1} v \triangle \lambda_v s_v - (n+1)^{-1} \sum_{v=1}^{n-1} \lambda_{v+1} s_v$$

$$+ (n+1)^{-1} n s_n \lambda_n - (n+1)^{-1} x_o \lambda_1$$

$$= T_{n1} + T_{n2} + T_{n3} + T_{n4},$$

say. In order to get (2.1), we are going to show that

$$\sum_{n=1}^{\infty} \left| \frac{\varphi_n}{n} T_{nr} \right|^p < \infty,$$

for $r = 1, 2, 3, 4$.

Now, appyling Hölder's inequality, we have

$$\sum_{n=2}^{m+1} \left| \frac{\varphi_n}{n} T_{n1} \right|^p \leq \sum_{n=2}^{m+1} \frac{|\varphi_n|^p}{n^{p+1}} \left\{ \sum_{v=1}^{n-1} v \triangle \lambda_v |s_v|^p \right\} \times$$

$$\left\{ \frac{1}{n} \sum_{v=1}^{n-1} v \triangle \lambda_v \right\}^{p-1}$$

$$= 0(1) \sum_{v=1}^{m} v \triangle \lambda_v |s_v|^p \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^p}{n^{p+1}}$$

$$= 0(1) \sum_{v=1}^{m} v \triangle \lambda_v \mid s_v \mid^p . \frac{|\varphi_v|^p}{v^{p-\mu}} \int_v^\infty \frac{dt}{t^{1+\mu}}$$

$$= 0(1) \sum_{v=1}^{m} v \triangle \lambda_v \left(\frac{|\varphi_v s_v|}{v} \right)^p$$

$$= 0(1) \sum_{v=1}^{m-1} \triangle (v \triangle \lambda_v) \sum_{k=1}^{v} \left| \frac{\varphi_k s_k}{k} \right|^p$$

$$+ 0(1) m \triangle \lambda_m \sum_{k=1}^{m} \left| \frac{\varphi_k s_k}{k} \right|^p$$

$$= 0(1) \sum_{v=1}^{m-1} v \triangle^2 \lambda_v \log v + 0(1) \sum_{v=1}^{m-1} \lambda_{v+1} \log_v$$

$$+ 0(1) m \triangle \lambda m \log m = 0(1)$$

as $m \to \infty$, by virtue of Lemmas 2 and 3.

Also,

$$\sum_{n=2}^{m+1} \left| \frac{\varphi_n}{n} T_{n2} \right|^p \leq \sum_{n=2}^{m+1} \frac{|\varphi_n|^p}{n^{2p}} \left| \sum_{v=1}^{n-1} \lambda_{v+1} s_v \right|^p$$

$$\leq \sum_{n=2}^{m+1} \frac{|\varphi_n|^p}{n^{2p}} \left\{ \sum_{n=1}^{n-1} \lambda_{v+1} |s_v|^p \right\} \times \left\{ \sum_{v=1}^{n-1} \lambda_{v+1} \right\}^{p-1}$$

$$= 0(1) \sum_{n=2}^{m+1} \frac{|\varphi_n|^p}{n^{p+1}} \sum_{v=1}^{n-1} \lambda_{v+1} |s_v|^p$$

$$= 0(1) \sum_{v=1}^{m} \lambda_{v+1} |s_v|^p \sum_{n=v+1}^{m+1} \frac{|\varphi_n|^p}{n^{p+1}}$$

$$= 0(1) \sum_{v=1}^{m} \lambda_{v+1} \left| \frac{\varphi_v s_v}{v} \right|^p$$

$$= 0(1) \sum_{v=1}^{m-1} \triangle \lambda_{v+1} \sum_{k=1}^{v} \left| \frac{\varphi_k s_k}{k} \right|^p + 0(1) \lambda_{m+1} \sum_{k=1}^{m} \left| \frac{\varphi_k s_k}{k} \right|^p$$

$$= 0(1) \sum_{v=1}^{m-1} \triangle \lambda_{v+1} \log v + 0(1) \lambda_{m+1} \log m = 0(1)$$

as $m \to \infty$, by virtue of Lemmas 1 and 2.

Similarly, using Lemma 1 and Lemma 4 we get

$$\sum_{n=1}^{m} \left| \frac{\varphi_n}{n} T_{n3} \right|^p \leq \sum_{n=1}^{m} \lambda_m{}^p \left| \frac{\varphi_n s_n}{n} \right|^p$$

$$= \sum_{n=1}^{m-1} \triangle \lambda_n{}^p \sum_{k=1}^{n} \left| \frac{\varphi_k s_k}{k} \right|^p + \lambda_m{}^p \sum_{k=1}^{m} \left| \frac{\varphi_k s_k}{k} \right|^p$$

$$= 0(1) \sum_{n=1}^{m-1} \triangle \lambda_n{}^p \log n + 0(1) \lambda_m{}^p \log m = 0(1).$$

Finally, we have

$$\sum_{n=1}^{m} \left| \frac{\varphi_n}{n} T_{n4} \right|^p = 0(1) \sum_{n=1}^{m} \frac{|\varphi_n|^p}{n^{2p}} = 0(1).$$

Therefore, we get

$$\sum_{n=1}^{\infty} \left| \frac{\varphi_n T_n}{n} \right|^p < \infty$$

which completes the proof of the theorem.

The special cases of this theorem for $\varphi_n = n^{1-p^{-1}}$ and $\varphi_n = n^{\gamma+1-p^{-1}}$ give, respectively, the following corollaries

COROLLARY 1: Let (λ_n) is a convex sequence such that $\Sigma\, n^{-1} \lambda_n$ is convergent. If the series $\Sigma\, x_k$ is $[R, \log n, 1]_p$ -bounded then $\Sigma\, \lambda_k x_k$ is $|C,1|_p$ -summable.

This results was proved by Mishra [3].

COROLLARY 2: Let (λ_n) be a convex sequence such that $\Sigma\, n^{-1} \lambda_n$ is convergent and $0 \leq \gamma\, p < 1$. If

$$\sum_{n=1}^{m} \frac{|s_n|^p}{n^{1-\gamma p}} = 0(\log m) \text{ as } m \to \infty$$

then $\Sigma\, \lambda_k x_k$ is $|C. 1; \gamma|_p$ -summable.

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3. ÖZET

Bu çalışmada şu teorem ispat edilmiştir: (λ_n) dizisi $\Sigma n^{-1}\lambda_n$ serisi yakınsak olacak şekilde konveks bir dizi olsun. Eğer $(n^{\mu-p}|\varphi_n|^p)$ dizisi artmayan olacak şekilde bir $\mu > 0$ sayısı mevcut ve Σx_k serisi $|B, \log n, \varphi, 1|_p$ -sınırlı ise $\Sigma\lambda_k x_k$ serisi φ-$|C, 1|_p$ toplanabilirdir.

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