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Chebyshev Approximation With Two Null Points

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Chebyshev Approximation With Two Null Points

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ABSTRACT

Chebyshev approximation problem with two null points, its best approximation and uniqueness are proved.

INDRODUCTION

Let CZ($[0, \alpha]$) be a space of functions which are continious in $[0, \alpha]$ and vanish at a null point. And let this space is made up by norm

$$\| g \| = \{ \sup |g(x)| : 0 \leq x \leq \alpha \}$$

If parameter space P is selected such as an open subspace of space R^n, then an element of P, has the form $A = (a_1, \ldots, a_n)$.

Whenever an element f of CZ($[0, \alpha]$) is handled then the existence of approximation function F which is an element of CZ($[0, \alpha]$) is accepted such that F (A,.) = F where A is an element of parameter space P.

The essential of Chebyshev approximation problem is to search an element A* of parameter space P, such that $e(A) = \| f - F(A,.) \|$ be minimum. Such a parameter A* is called the best parameter and F(A*,.) the best approximation to f.

Chebyshev approximation with a null point is studied by C.B. Dunham [1] in 1972. Dunham presented "DE LA VALLE POUSSIN" type result without proof, wich is useful in characterizing "near best approximation". In addition, Dunham said that the existence of Chebyshev approximation with a null point

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and its uniqueness without proof, because conception of local Haar property is not sufficient for the theorem which shows uniqueness of best approximation.

In this study, the cases in which functions be zero at 0 and α are investigated by new concepts. The theorems which are not proved by Dunham are presented here with proofs and "Chebyshev approximation problem with two null points" is solved.

CHEBYSHEV APPROXIMATION WITH TWO NULL POINTS

Let $C([0, \alpha])$ be space of continious functions with real value on $[0, \alpha]$, define norm.

$$\| g \| = \{\sup |g(x)| : 0 \leq x \leq \alpha\}$$

for the space.

It is casily seen that linear space

$$CZ_o([0, \alpha]) = \{ f : f \in C(0, \alpha) : f(0) = 0\}$$

and

$$CZ([0, \alpha]) = \{g : g \in CZ_o(0, \alpha) : g(\alpha) = 0\}$$

are closed in $C([0, \alpha])$.

Let R^n a non-empty subset of P which is a parameter space. Suppose the existence of approximation function F which is element of $CZ([0, \alpha])$ and obey the relation $F(A,.) = F[A] = F$ for element f of P.

If there exist any element f of space $CZ([0, \alpha])$ then, finding an element A of P, which makes

$$e(A) = \| F(A,.) - f \|$$

minimum will be the essential of Chebyshev approximation problem. More clearly, we can write it such as the following

$$\begin{aligned} \rho(f) &= \inf \{ \| F(A,.) - f \| : A \in P\} \\ &= \| F[A^*] - f \| \end{aligned}$$

Such a parameter A is called the "best parameter" and function $F(A^*,.)$ "best approximation to f"

Definition 1 Suppose a subset, $\{1, \varnothing_1, \dots, \varnothing_n, \dots\}$ of $C([0, \alpha])$. Let be $c_o, c_1, \dots, c_k \in IR$ and call

$$p_k = \sum_{i=0}^{k} c_i \varnothing_i$$

If for each element f of $C([0, \alpha])$ and each number $\epsilon > 0$ correspond to a natural number k such that

$$\| f - p_k \| \leq \epsilon$$

then set $\{1, \varnothing_1, \varnothing_2, ...\}$ is called "spanner" in $C([0, \alpha])$.

Theorem 1 If the set $\{1, \varnothing_1, ..., \varnothing_n, ...\}$ is a spanner in $C([0, \alpha])$ and

$$0 = \varnothing_1(0) = \varnothing_2(0) = ... \qquad (1)$$
$$0 = \varnothing_1(\alpha) = \varnothing_2(\alpha) = ... \qquad (2)$$

are fulfilled, then the following properties are exist:

1) $CZ_o([0, \alpha]) = CZ([0, \alpha])$

2) Set $\{\varnothing_1, \varnothing_2, ..., \varnothing_n, ...\}$ is a spanner in $CZ([0, \alpha])$.

Proof 1 Let the set $\{1, \varnothing_1, \varnothing_2, ..., \varnothing_n, ...\}$ be a spanner in $C([0, \alpha])$ and conditions be fulfilled.

Suppose $f \in CZ_o([0, \alpha])$. So $f \in C([0, \alpha])$, then for each $\epsilon > 0$ there are real numbers $c_o, c_1, ..., c_k$ such that

$$\| f - p_k \| \leq \epsilon \qquad (3)$$

Equation (3) gives us property

$| f(x) - (c_o + c_1\varnothing_1(x) + c_2\varnothing_2(x) + ... + c_k\varnothing_k(x)) | \leq \epsilon$
for each $x \in [0, \alpha]$. The fact that $f(0) = 0$ gives $| c_o | \leq \epsilon$ and the inequality gives rise to $c_o = 0$. As a result, we get relation
$| f(x) - (c_1\varnothing_1(x) + c_2\varnothing_2(x) + ... + c_k\varnothing_k(x)) | \leq \varepsilon$

for each $x \in [0, \alpha]$. If we put α instead of x, for each $x > 0$ in the above relation we get $f(\alpha) = 0$ from the fact $| f(\alpha) | \leq \epsilon$. Therefore

$$CZ_o([0, \alpha]) = CZ([0, \alpha])$$

Proof 2 On the other hand, if we use the proof technique above mentioned, can be easily seen that the set $\{\varnothing_1, \varnothing_2, ..., \varnothing_n, ...\}$ be a spanner in $CZ([0, \alpha])$.

Definition 2 If there exist point set $\{x_1, x_2 ..., x_m\}$ such that $0 < x_1 < x_2 < ... x_m < \alpha$ and

$\| g \| = | g(x_i) |, \; g(x_i) = (-1)^i g(x_1); \; i = 1,2, \ldots m$
for an element g of CZ([0, α]) then g alternates m times in [0, α]. If function g alternates m times but not m+1 times then we will say that g alternates definitely m times.

Definition 3 Let $A = (a_1, a_2, \ldots, a_n)$ and F(A,.) be an approximation function of CZ([0, α]). Let $i = 1, 2, \ldots, n$ and vector space D(A,.,.) with dimention m which is made up of $\partial F(A,.) / \partial a_i$. Evidently $m \leq n$.

If the condition F(A,.) = F (B,.) requires that F(A,.) - F(B,.) has m null points at [0,α] then approximation function F has property (CZ).

Now let us try to have result DE LA VALLE-POUSSIN type.

Theorem 2 If an approximation function F(A,.) of space CZ([0,α]) has property (CZ) and function f is an element of CZ([0,α]) such that function F(A,.) - f alternates definitely m+1 times then each element B of parameter space P has the following property

$$\text{Max } \{ | F(B, x_i) - f(x_i) | \; : \; i = 0, \ldots, m \} \geq$$
$$\geq \text{Min } \{ | F(A,x_i) - f(x_i) | \; : i = 0, \ldots, m \} \qquad (4)$$

Proof

1) Relation

$$(F(A,x_i) - f(x_i)) (F(A,x_i) - F(B,x_i)) \leq 0 \qquad (5)$$

is exist, otherwise

$$(F(A ,x_i) - f(x_i)) (F(A,x_i) - F(B,x_i)) > 0 \qquad (6)$$

would be true. Function F(A,.) - f alternates on space $\{x_o, x_1, \ldots, x_m\}$ Accoding to Equation (6) it is clear that function $F(A,x_i)-F(B,x_i)$ will have the same alternant. Let us assume $F(A,x_o) - f(x_o) > 0$ without violating the generality, so that the relations

$$F(A,x_o) - F(B,x_o) > 0$$
$$F(A,x_1) - F(B,x_1) < 0$$

. . . .

prove that $F(A,.) - F(B,.)$ has m zeros. This requires $F(A,.) = F(B,.)$ in accordance with the property (CZ). This result contrasts the assumtion (6).

2) $\underset{0 \le p \le m}{\mathrm{Max}} \{ | F(B,x_i)-f(x_i) | \; ; i=0,1,\ldots,m \} = | F(B,x_p)-f(x_p) | \ge$

$\underset{0 \le q \le m}{\mathrm{Min}} \{ | F(A,x_i) - f(x_i) | \; ; i=0,1,\ldots,m \} = | F(A,x_q) - f(x_q) |$

If the relation (4) was correct then we would have found

$$| F(B,x_p) - f(x_p) | < | F(A,x_q) - f(x_q) | \qquad (7)$$

It is clear that the inequality (7) is true for each x_i: $i = 0,1,\ldots,m$ and then we have

$$| F(B,x_i) - f(x_i) | < | F(A,x_i) - f(x_i) | \qquad (8)$$

If we add $f(x_i)$ to the second factor of relation (5) we find,

$$(F(A,x_i) - f(x_i)) \, ((F(A,x_i) - f(x_i)) - (F(B,x_i) - f(x_i))) \le 0 \qquad (5')$$

Function $F(A,.) - f$ alternates on set $\{x_o, x_1, \ldots, x_m\}$ so we can take $F(A,x_o) - f(x_o) > 0$ without violating the generality.

Under these circumtances from the inequality (5′) we can find

$$(F(A,x_o) - f(x_o)) \le (F(B,x_o) - f(x_o))$$

We know that $F(A,x_o) - f(x_o) > 0$, so $F(B,x_o) - f(x_o)$ is also positive. Hence we have

$$| F(A,x_o) - f(x_o) | \le | F(B,x_o) - f(x_o) |$$

However, this relation contrasts to the equation (8).

Definition 4 Let an element f of space $CZ([0,\alpha])$ and $F(A,.)$ be best to f. For the best approximation to f necessary and sufficient condition is that $F(A,.)$ alternates definitely $m+1$ times then it is said that $F(A,.)$ has property (CS) with degree $m+1$

Deifinition 5 Let $A = (a_1, a_2, \ldots, a_n)$ and $F(A,.)$ be an approximation function at $CZ([0,\alpha])$ and if the following condutions exist then approximation $F(A,.)$ has local Haar property with null points m degree at A.

(i) $\partial F(A,.) / \partial a_i$ are exist and continious for each i; $i = 1,2, \dots, n$

(ii) Let

$$B = (b_1,b_2, \dots, b_n) \text{ and } D(A,B,x) = \sum_{i=1}^{n} b_i \frac{\partial F(A,x)}{\partial a_i}$$

When $\| B \|$ is sufficientyl small and relation

$$F(A-B,x) - F(A,x) = D(A,B,x) + R(A,B,x)$$

is correct, then

$$R(A,B,x) = O \| B \|$$

(iii) Element A has a neighbourhood which is contained by P.

(iV) Linear space $D(A,.)$ is an Haar space with dimention m at $[0, \alpha]$.

Theorem 3 If approximation function F of space $CZ([0, \alpha])$ has local Haar property at A and $F(A,.)$ be best to element f of $CZ([0, \alpha])$ then in that case F has property (CS) with degree $m+1$.

Proof Suppose F has not property (CS) with degree $m+1$. Now, function $F(A,.)-f$ alternates at $[0, \alpha]$ less than $m+1$. There is relation

$$F(A,0) - f(0) = 0$$

So there is a number γ such that

$| F(A,x) - f(x) | < e(A)/2 \ : \ 0 \leq x \leq \gamma \leq \alpha$

where $e(A) = \| F(A,.) - f \|$

At the point α, we can write

$$F(A,\alpha) - f(\alpha) = 0$$

So there exist $\delta > 0$ such that

$| F(A,x) - f(x) | < e(A)/2 \ ; \ \delta \leq x \leq \alpha$

If approximation function $F(A,.)$ is best to f at $[0,\alpha]$ then $F(A,.)$ will be best to f at the interval $[\gamma,\delta]$ ([2], Theorem 3)

Under these circumtances of the function $F(A,.) - f$ alternates $m+1$ times at $[0,\alpha]$ then $F(A,.) - f$ also alternates $m+1$ times at the interval $[\gamma,\delta]$ ([3], Theorem 5).

If the function F(A,.)–f alternates less than $m+1$ times we find an element B of parameter space P with the help of Theorem 2, under the condition $\gamma \leq x \leq \delta$, such that

$$\| F(B,.) - F(A,.) \| < e(A)/2$$
$$| F(B,x) - f(x) | < e(A)$$

Then we have

$$| F(B,x) - f(x) | \leq | F(B,x) - F(A,x) | + | F(A,x) - f(x) |$$
$$< e(A)/2 + e(A)/2$$
$$< e(A)$$

From this and the fact that $e(B) = \| F(B,.) - f \|$ we have $e(B) < e(A)$. However this contrasts to the best approximation. As a rĕsult, function F(A), f alternates definitely $m+1$ times.

Combining Theorem 2 and 3 we get the following result:

Theorem 4 If approximation function F(A,.) has the property (CZ) and local Haar property with null points degree m, then, necessary and sufficient condition for approximation function F(A,.) being best to f is that F(A,.) has property (CS) with degree $m+1$.

Theorem 5 Let approximation function F(A,.) has the condition of Theorem 4. If F(A,.) is best it will be unique.

Proof Let us choose γ and δ, $\gamma \leq x \leq \delta$, such that

$$| F(A,x) - f(x) | < e(A)/2$$

If approxation function F(A.,) be best to f at the interval $[0, \alpha]$, F(A.,) is also best to f at $[\gamma, \delta]$ ([2] ; Theorem 3)

In that case, if function F(A.,) — f alternates $m+1$ times at interval $[0,\alpha]$ and also alternates $m+1$ times at $[\gamma,\delta]$ ([3]; Theorem 5)

So it is sufficient to prove the theorem at interval $[\gamma,\delta]$.

Let us suppose F(A,.) is not the best and unique approximation to f at the interval $[\gamma,\delta]$, take two approximation functions as F (A,.) and F(B,.).

Function F(A,.) – f alternates $m+1$ times at the interval $[\gamma,\delta]$ so there is the property:

$$F(A,x_i) - f(x_i) = -(F(A,x_{i-1}) - f(x_{i-1})) \quad ; \quad i = 0,1,\dots,m$$

In accordance with the Theorem 2 we get

$$F(A,x_o) - F(B,x_o) \leq 0$$
$$F(A,x_1) - F(B,x_1) \geq 0$$
$$\dots\dots\dots\dots\dots\dots\dots$$

If there exist definite inequality, then function $F(A,.) - F(B,.)$ will have definite $m+1$ null points at $[\gamma,\delta]$ and from Haar condition one can get equation

$$F(A,.) = F(B,.)$$

$D(B,.,.)$ fulfills Haar condition so $F(B,.)$ has local Haar property in the case of equality.

Let $0 \leq t \leq 1$ and C be an element of parameter space P, if we apply property (ii) of local Haar condition to function

$$F(B,x) - F(B-tC,x)$$

we will get

$$F(A,x) - F(B-tC,x) = F(A,x) - F(B,x) + tD(B,C,x) + R(B,C,x)$$

If we choose t sufficienlty small we will have system

$$F(A,x_o) - F(B-tC,x_o) < 0$$
$$F(A,x_1) - F(B-tC,x_1) > 0$$
$$\dots\dots\dots\dots\dots\dots\dots$$

Function $F(A,.) - F(B-tC,.)$ has m null points at the interval, and while t approaching zero, $F(A,.) = F(B,.)$ equality will be exist.

Here, function $F(A,.)$ is the best unique approximation to f at the interval $[\gamma,\delta]$. If there had been another best approximation at $[0,\alpha]$ it would have also been best approximation at $[\gamma,\delta]$, thet is impossible.

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ÖZET

İki noktada sıfır olan Chebyshev yaklaşım problemi, en iyi yaklaşımı ve bunun tekliği ispatlanmaktadır.

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