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by

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Absolute Convexity In The Spaces Of Strongly σ– Summable Sequences

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ABSTRACT:

Nanda has considered strongly almost summable sequences and introduced the ronvexity in strongly almost summable sequences. Maddox and Roles have introduced the absolute convexity in certain topological linear spaces. The purpose of this paper is to introduce the absolute convexity in strongly σ – summable sequences.

1. Introduction:

Let l_∞, c, c_0 be the Banach spaces of bounded, convergent and null sequences $x = \{x_k\}$ with usual norm $\|x\| = \sup_k |x_k|$. Let σ – be a mapping of the set of positive integers into itself. A continuous linear function $\varnothing$ on l_∞ is said to be an invariant mean or a σ – mean if and only if (i) $\varnothing(x) \geq o$ when $x_n \geq o$ for all n (ii) $\varnothing(e) = 1$, where $e = \{1,1,1...\}$, and (iii) $\varnothing\{x_{\sigma(n)}\} = \varnothing(x)$ for each $x \varepsilon l_\infty$ Throughout this paper we deal only with mappings σ – which are one to one such that $\sigma^m(n) \neq n$ for all m and n, where $\sigma^m(n)$ is the m^{th} iterate of σ at n. For such mappings, every σ – mean extends the limit functional on c (see Raimi [6]), in the sense that $\varnothing(x) = \lim x$ for all $x \varepsilon c$. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ – means are equal.

The strongly summable sequences have been systematically investigated by Hamilton and Hill [2], Kuttner [1] and some others. The spaces of strongly summable sequences were introduced and studied by Maddox ([3], [4]). Maddox and Roles [5]

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[5] have introduced the absolute convexity in certain topological spaces. Nanda [8] has considered strongly almost summable sequences and introduced the r – convexity in strongly almost summable sequences. More recently Saraswat and Gupta [7] has considered strongly σ – summable sequences.

The purpose of this paper is to introduce the absolute convexity in strongly σ – summable sequences, which will fill up a gap in the existing literature.

2. Preliminaries: Let $A = (a_{nk})$ be an infinite matrix of non-negative real numbers and p_k is real such that $p_k > 0$ and $\sup_k p_k < \infty$. If $x = \{x_n\}$, write $Tx = \{x_{\sigma(n)}\}$. It is easy to show that V_σ can be characterized as the set of all bounded sequences x for which $\lim_m (x + Tx + ... + T^m x)/(m+1)$ exists in l_∞ and has the form Le where $L = \sigma - \lim x$. Throughout this paper we shall use the notation a (n,k) to denote the element a_{nk} of the matrix A for which $m \geq 0$, we have

$$(Ax + TAx + ... + T^m Ax) / (m+1)$$
$$= \{\sum_k [a(n,k) + a(\sigma(n),k) + ... + a(\sigma^m(n),k)] x_k / (m+1)\}_{n=1}^{\infty}$$

where $\sigma^m(n)$ denotes the m^{th} iterate of σ at n. We write for all integers m, $n \geq 1$

$$T_{m,n}(x) = \sum_{k=1}^{\infty} \sum_{j=0}^{m} a(\sigma^j(n), k) |x_k|^{p_k} / (m+1)$$
$$= \sum_k \alpha(n,k,m) |x_k|^{p_k},$$

where

$$\alpha(n,k,m) = \frac{1}{m+1} \sum_{j=0}^{m} a(\sigma^j(n), k).$$

We now write (see [7])

$$[A_\sigma, p]_o = \{x ; T_{mn}(x) \to o \text{ uniformly in } n\},$$

and

$$[A_\sigma, p]_\infty = \{x : \sup_{mn} T_{mn}(x) < \infty\}.$$

The sets $[A_\sigma,p]_o$ and $[A_\sigma,p]_\infty$ will be respectively called the spaces of strongly σ – summable to zero and strongly σ – summable bounded sequences.

If we take $\sigma(n) = n + 1$, then these spaces reduced to spaces of strongly almost summable to zero and strongly almost bounded sequences respectively (see [8]).

In this paper we study absolute convexity and locally boundedness. We start with some definitions..

For $o < r \leq 1$ a non-void subset W of a linear space is said to be r-convex if x, y$\in$W and $|\lambda|^r + |\mu|^r \leq 1$ together imply that $\lambda x + \mu y \in W$. It is clear that if W is absolutely r - convex, then it is absolutely p-convex for $p < r$. A linear topological space X is said to be r - convex if every nbd of $0 \in X$ contains an absolutely r-convex nbd of $0 \in X$. The r - convexity for $r > 1$ is of little interest, since X is r – convex for $r > 1$ if and only if X is the only neighbourhood of $0 \in X$ (see Maddox and Roles [5]). A subset Y of X is said to be bounded if for each neighbour hood W of $0 \in X$ there exists an integer $N > 1$ such that $Y \subseteq NW$, X is called locally bounded if there is a bounded neighbourhood of zero. Also, we define natural distance function

$$h(x) = \sup_{m,n} (T_{m,n}(x))^{1/M}$$

($\inf p_k > 0$), where $M = \max(1, \sup p_k = H)$, and for r-convexity we define $s(n) = \{k: 0 < \alpha(n,k,m), \sup_m \alpha(n,k,m) < \infty \text{ and } p_k < r\}$ for $r > 0$.

3. We first state a number of useful inequalities.

Inequality: Let x,y,λ,μ be complex numbers.
Then

(i) $0 < p \leq 1$ implies
$$|x + y|^p \leq |x|^p + |y|^p.$$

(ii) $p \geq 1$ and $|\lambda| + |\mu| \leq 1$ imply
$$(|\lambda x| + |\mu y|)^p \leq |\lambda| |x|^p + |\mu| |y|^p.$$

(iii) $|x| \leq 1$, $0 < p < r$ and $N > 1$ imply
$$|x|^p < |x|^r(1 + N \log N) + N^\pi,$$

where $\dfrac{1}{\pi} + \dfrac{r}{p} = 1$; and π is the conjugate of p.

Proof: (i) is well known. A proof of (ii) is given in [5] and (iii) is a slightly generalization of a result used in [9].
We now prove

Theorem 1. Let $[A_\sigma, p]_\infty$ be a paranormed space, let $p \in l_\infty$, $o < r \leq 1$ and suppose that there exists an integer $N > 1$ such that

$$\sup_{m,n} \sum_{s(n)} N^{\pi_k} < \infty, \tag{3.1}$$

where $1/\pi_k + r/p_k = 1$. Then $[A_\sigma, p]_\infty$ is r-convex.

Proof. Consider an absolutely r-convex set $W(d)$ containing the origin $\theta = (o,o,o,\ldots)$ to show that $W(d)$ form a nbd base of θ (for $o < d \leq 1$).

We define $q_k = \max(r, p_k)$ $\forall$ k and $\forall$ $d > 0$, also

(a) $W_1(d) = \{x \in [A_\sigma, p]_\infty : \sup_{m,n} \Sigma \quad (\alpha \quad (n,k,m) \quad |x_k|^{p_k})^{q_k/p_k} \leq d\}$

(b) $W_2(d) = \{x \in [A_\sigma, p]_\infty : \quad \sup_{m,n} (\alpha \quad (n,k,m) \quad |x_k|^{p_k} \leq d\}$,

and $W(d) = W_1(d) \cap W_2(d)$. Now if $x, y \in W(d)$ and $|\lambda|^r + |\mu|^r < 1$, then $|\lambda| + |\mu| \leq 1$. We have

$(|\lambda x_k| + |\mu y_k|)^{q_k} \leq |\lambda|^r |x_k|^{q_k} + |\mu|^r |y_k|^{q_k}|$, for $q_k < 1$ and $q_k \geq 1$

(using inequalities (i) and (ii))

whence $x, y \in W_1(d) \Rightarrow \lambda x + \mu y \in W_1(d)$. Also, since $x, y \in W_2(d)$ and $|\lambda| + |\mu| \leq 1$, it is easy to see that $\lambda x + \mu y \in W_2(d)$. Therefore $W(d)$ is an absolutely r-convex set containing θ. (It may be noted here that $p \in l_\infty$ is taken for the linearity of $[A_\sigma, p]_\infty$ (see [7])).

Let $S(R)$ be the sphere of centre θ and radius $R > 0$ i,e. the set of all $x \in [A_\sigma p,]_\infty$ such that $h(x) \leq R$. Now, it is easy to prove that $W(d) \supset S(d^{1/M})$ for $o < d \leq 1$, so that $W(d)$ is a nbd of θ

Finally, we prove that for each $\varepsilon > o$, there is a $d = d(\varepsilon) > o$ such that $o < d \leq 1$ and $W(d) \subset S(\varepsilon)$. Let $t(n)$ be the set of all $k \in s(n)$ such that $p_k < r/2$. By (3.1) we see that $t(n)$ is a finite set $\forall n$. Let $N(n)$ be the number of integers in $t(n)$, Therefore, we have

$$\sum_{s(n)} N^{\pi_k} \geq \sum_{t(n)} N^{-1} = N^{-1}N(n) \tag{3.2}$$

whence

$$H' = \sup N(n) < \infty . \tag{3.3}$$

Now, let $x \in W(d)$ for some d with $o < d \leq 1$. We take h (x) by splitting $T_{m,n}(x)$ as follows,

$$T_{m,n}(x) = \Sigma \alpha (n,k,m) |x_k|^{p_k}$$

$$= \sum_1 \text{ over } p_k \geq r + \sum_2 \text{ over } p_k < r/2 + \Sigma_3 \text{over } r/2 \leq p_k < r.$$

$$= I_1 + I_2 + I_3, \text{ say,} \tag{3.4}$$

Now, for each $n \geq 1$,

$$I_1 = \sum_1 \alpha (n,k,m) | x_k |^{p_k} = \Sigma_1 (\alpha (n,k,m) |x_k|^{p_k})^{q_k/p_k} \leq d, \tag{3.5}$$

since $x \in W_1(d)$ and $p_k = q_k$ when $p_k \geq r$.

Also

$$I_2 = \sum_2 \alpha (n,k,m) |x_k|^{p_k} \leq d. H', \text{ for } p_k < r/2 \tag{3.6}$$

(using (b) and (3.3.)

and

$$I_3 = \sum_3 \alpha (n,k,m) |x_k|^{p_k}$$

$$\leq (1+N \log N) \sum_3 (\alpha (n,k,m) |x_k|^{p_k})^{q_k/p_k} + \Sigma_3 N^{\pi_k} \tag{3.7}$$

(by inequality (iii))

as we have $q_k = r$ for $r/2 \leq p_k < r$. (for each $N > 1$ and $n \geq 1$).
Now let R be a positive integer. If $r/2 \leq p_k < r$ then $\pi_k \leq -1$, whence

$$\sum_3 (RN)^{\pi_k} \leq R^{-1} \sum_{s(n)} N^{\pi_k},$$

so that $\sup_{m,n} \sum_3 N^{\pi_k}$ can be made arbitrarily small by a choice of a suitably large N. Therefore

$$\sup_{m,n} \sum_2 N^{\pi_k} < (\varepsilon/2)^M (\text{for } \varepsilon > o, N > 1) \tag{3.8}$$

and $o < d \leq 1$ such that

$$d(2 + H' + N \log N) < (\varepsilon/2)^M.$$

Thus, by using Inequality (i), (3.5), (3.6), (3.7) and (3.8) we have

$h(x) < \varepsilon$, whenever $x \in W(d)$ for $M \geq 1$.
Hence $W(d) \subset S(\varepsilon)$.

This completes the proof.

Theorem 2. Let $S = \{k: o < \sup_m \alpha(n,k,m) < \infty\}$ and $T = \{k: k \in S$ and $\alpha(n,k,m) \to o\ (n \to \infty)\}$. Let $[A_\sigma, p]_\infty$ (respectively $[A_\sigma, p]_o$) be paranormed. Then it is locally bounded if and only if $\inf_S p_k > o$ (respectively $\inf_T p_k > 0$).

Proof: Necessity. Suppose that $[A_\sigma, p]_\infty$ is locally bounded. Then, there is a bounded nbd B of θ such that $S(d) \subset B$ for $d > o$. Since B is bounded there is a non zero β such that

$$\beta\, S(d) \subset \beta\, B \subset S(d/2).$$

We define $x^{(k)} \in S(d)$, by

$$x^{(k)} = (d^M/\sup_{m,n} \alpha(n,k,m))^{1/p_k}\, e^{(k)}.$$

Then

$$h(\beta x^{(k)}) = d\ |\beta|^{p_k/M} \leq d/2,$$

Therefore

$$\inf_S p_k > 0.$$

Sufficiency: We shall show that the sphere S (1) of centre θ and radius 1 is a bounded nbd of θ. Let N be any nbd of θ. Then, there is a sphere $S(d) \subset N$. If $x \in S(1)$, then

$$h(x/\beta) = \sup_{m,n}(\sum_S \alpha(n,k,m)\ |x/\beta|^{p_k})^{1/M} (\text{for } |\beta| \geq 1)$$

where

$$M = \max(1, \sup_s p_k).$$

Now, choose β such that $|\beta| \geq 1$ and

$$|\beta|^{-\inf_s p_k} < d^M.$$

Then

$$h(x/\beta) \leq |\beta|^{-\inf_s p_k/M}$$

Therefore, $x/\beta \in S(d) \subset N$, so that $S(1) \subset \beta N$, i.e. $S(1)$ is a bounded nbd of θ.

The proof for $[A_\sigma, p]_o$ is similar.

This terminates the proof.

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ÖZET

Nanda kuvvetli hemen hemen toplanabilen dizileri incelemiş ve kuvvetli hemen hemen toplanabilen dizilerde r-konveksliği tanımlamıştır. Maddox ve Roles ise bazı topolojik lineer uzaylar da mutlak konvekslik kavramını vermişlerdir. Bu çalışmanın amacı kuvvetli σ-toplanabilen dizilerde mutlak konvekslik kavramını tanımlamaktır.

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