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Linear Restrictions on Parameters in Linear Regression Model With non-spherical Disturbances

by

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Linear Restrictions on Parameters in Linear Regression Model With non-spherical Disturbances

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ABSTRACT

In this paper we are concerned with the estimation of parameters subject to linear constraints. No rank conditions are imposed on R,X and V. The results are applied to obtain the generalized inverse of $X'V^{+}X$ which yields a solution of the normal equations subject to nonestimable constraints on the parameters.

1. INTRODUCTION

Consider the linear model

$$Y = X\beta + u, \quad u \sim N(0, \sigma^2 V) \tag{1.1}$$

where Y is Nx1, X an Nxk matrix of fixed quantities, and β a kx1 vector of unknown parameters, u is an Nx1 vector of jointly normal disturbances with mean vector zero and nonnegative definite covariance matrix V. Further suppose that one is contemplating appending the model with $m < k$ independent linear restrictions on β,

$$R\beta = r \tag{1.2}$$

where R is an mxk matrix of known constants, and a r known kx1 vector. The equations $R\beta = r$ are assumed to have at least one solution in β.

2. MATRIX RESULTS

In this section we establish some of the matrix results, we shall use in the sequel. In particular, we shall establish those properties of the generalized inverse. We write A^{-}(nxm) to denote a generalized inverse or g-inverse of a matrix A (mxn), that is, any matrix A^- satisfying $A A^- A = A$ and we write A^+ to denote the Moore-Penrose inverse, defined as the unique matrix G satisfying each of the following four conditions:

$$A\,G\,A = A \quad (1)$$
$$G\,A\,G = G \quad (2)$$
$$(AG)' = AG \quad (3)$$
$$(GA)' = GA \quad (4)$$

The following notation and results will be used throughout the paper.

$$Q_A = I - A^+A$$
$$(A')^+ = (A^+)'$$
$$(A'A)^+ = A^+(A^+)'$$
$$A'(AA')^+ = (A'A)^+A' = A^+$$
$$A\,(A'A)^+A'A = A$$

$R\,R^+r = r$ provided $R\,\beta = r$ are consistent.

$$(XQ_R)' = Q_RX'$$
$$Q_R[Q_RX'V^{-1}XQ_R]^+Q_R = (Q_RX'V^{-1}Q_R)^+$$

If P is orthogonal matrix, then $P^+ = P'$.

3. THE BEST LINEAR ESTİMATE OF β WITH A POSITIVE DEFINITE V.

Now, consider the general linear model (1.1) with β subjected to the constraint (1.2). It is well known that every β satisfying (1.2) is specified by

$$\beta = R^+r + (I - R^+R)\,w \tag{3.1}$$

where w is an arbitrary vector of appropriate dimensions. Substituting (3.1) into (1.1) for β, we obtain

$$Y = X\,R^+r + X\,(I - R^+R)\,w + u$$

or

$$Y - X\,R^+r = X\,Q_R\,w + u. \tag{3.2}$$

Therefore, the best linear estimate (BLE) of w in (3.2) is given by

$$\hat{w} = [(XQ_R)'V^{-1}XQ_R]^{+} (XQ_R)'V^{-1} (Y - X R^{+}r). \tag{3.3}$$

Substituting (3.3) into (3.1) for $\hat{w}$, we get

$$\tilde{\beta} = R^{+}r + Q_R \hat{w}$$

and

$$\tilde{\beta} = R^{+}r + Q_R[Q_RX'V^{-1}XQ_R]^{+}Q_RX'V^{-1}(Y - X R^{+}r)$$

or

$$\tilde{\beta} = R^{+}r + (\tilde{X}Q_R)^{+}(\tilde{Y} - \tilde{X} R^{+}r). \tag{3.4}$$

where $\tilde{X} = V^{-1/2}X$ and $\tilde{Y} = V^{-1/2}Y$.

The problem we consider is the estimation of linear function of β by means of linear functions of Y. We use the following preliminary results.

LEMMA 3.1 There is a $\bar{\beta}$ of the form

$$\bar{\beta} = AY + c$$

such that $E(p'\bar{\beta}) = p'\beta$ for satisfying the consistent equations $R\beta = r$ if and only if there are vectors δ and ρ such that $p' = \delta'X + \rho'R$.

Proof: The proof of this lemma is given by Rao and Mitra (1971) and Gerig and Gallant (1975).

COROLLARY 3.1 Consider the model (1.1) subject to the linear equality constraints $R\beta = r$. The best linear unbiased estimator of $p'\beta$ is $p'\beta = p'R^{+}r + p'(\tilde{X} Q_R)^{+} (\tilde{Y} - \tilde{X}R^{+}r)$, where

$$p' = \delta'X + \rho'R.$$

4. SQUARE ROOTS OF POSITIVE SEMI-DEFINITE MATRICES

Consider the positive semi-definite matrix C. Let it be diagonalized as follows: $T'C\,T = D$, where T is orthogonal, Graybill (1969 p. 19). Any matrix B such that $B^2 = C$ will be called a square root of C. Since $C = T D^{1/2} D^{1/2} T' = T D^{1/2} T'T D^{1/2} T'$, obviously $T D^{1/2} T'$ will be such a square root, where $D^{1/2}$ is a diagonal matrix having the square roots of the corresponding diagonal elements of D as diagonal elements. Let C^+ be the Moore-Penrose inverse of C. Then C^+ can be written as TD^+T', where D^+ is a diagonal matrix having the reciprocal values of the corresponding nonzero diagonal elements of D and zeros as diagonal elements. The matrix $(C^{1/2})^+ = T (D^+)^{1/2} T'$ will then be a square root of C^+. Since

$$(C^{1/2})^+ = (TD^{1/2}T')^+ = T (D^{1/2})^+ T' = T (D^+)^{1/2} T'$$

we find that $(C^{1/2})^+ = (C^+)^{1/2}$. When C is positive definite we have

$$C^{1/2} = T D^{1/2} T'$$

and

$$C^{-1/2} = T D^{-1/2} T'.$$

5. SOLVING NORMAL EQUATIONS SUBJECT TO NON ESTIMABLE CONSTRAINTS

In this section, when $\sigma^2 V (\neq \sigma^2 I)$ is non-negative matrix, we would like to extend the theorem, given by Gerig and Gallant (1975)

Consider the problem of solving the normal equations

$$X'V^{-1} X \beta = X'V^{-1} Y \quad (\text{or } X'V^+X\beta = X'V^+Y)$$

subject to the non-estimable parametric constraints

$$R \beta = 0.$$

By non-estimable, we mean that there do not exist non-zero vectors δ and ρ such that

$$\delta' V^{-1/2} X = \rho' R, \quad (\text{or } \delta'(V^{1/2})^+ X = \delta' R$$

Our solution consists of showing that

$C_1 = (Q_R X'V^{-1}X Q_R)^+$ and $C_2 = (Q_R X'V^+X Q_R)^+$ are generalized inverses of $X V^{-1}X$ and $X'V^+X$ respectively.

THEOREM 5.1 If there do not exist non-zero vestors δ and ρ such that $\delta'(V^{1/2})^+X = \rho'R$, then

$$C_2 = (Q_R X'V^+X Q_R)^+$$

is a generalized inverse of $X'V^+X$.

Proof: Let

$$U = \begin{bmatrix} I & \vdots & -(V^{1/2})^{+}XR^{+} \\ \cdots & \vdots & \cdots\cdots \\ 0 & \vdots & I \end{bmatrix}, \quad Z = \begin{bmatrix} (V^{1/2})^{+}XQ_R \\ \cdots\cdots \\ R \end{bmatrix} = U \begin{bmatrix} (V^{1/2})^{+}X \\ \cdots \\ R \end{bmatrix}$$

Since U is non-singular, the ranks of Z and $\begin{bmatrix} (V^{1/2})^{+}X \\ \cdots\cdots \\ R \end{bmatrix}$ are the same. Since the linear spaces generated by the rows of $(V^{1/2})^{+}X$ and R are disjoint by hypothesis we have the

rank $\begin{bmatrix} (V^{1/2})^{+}X \\ \cdots\cdots \\ R \end{bmatrix}$ = rank $[(V^{1/2})^{+}X]$ + rank (R). Consider

$$\begin{bmatrix} \tilde{X} \\ \cdot\cdot \\ R \end{bmatrix} (Q_R \vdots R') = \begin{bmatrix} \tilde{X}Q_R & \vdots & \tilde{X}R' \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & RR' \end{bmatrix},$$

where $\tilde{X} = (V^{1/2})^{+}X$. Since $(Q_R \vdots R')$ and U have full rank, we write

$$\text{rank }(Z) = \text{rank} \begin{bmatrix} \tilde{X} \\ R \end{bmatrix} = \text{rank} \begin{bmatrix} \tilde{X}Q_R & \vdots & \tilde{X}R' \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & RR' \end{bmatrix},$$

$$= \text{rank } \left\{ U. \begin{bmatrix} \tilde{X}Q_R & \cdot & \tilde{X}R' \\ & \cdots\cdots & \\ 0 & \cdot & RR' \end{bmatrix} \right\} = \text{rank} \begin{bmatrix} \tilde{X}Q_R & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots \\ 0 & \vdots & RR' \end{bmatrix},$$

$$= \text{rank }(\tilde{X}Q_R) + \text{rank }(RR'),$$

$$= \text{rank }\{(V^{1/2})^{+}XQ_R\} + \text{rank }(R).$$

Thus, rank $\{(V^{1/2})^{+}XQ_R\}$ = rank $\{(V^{1/2})^{+}X\}$. Since the columns of $(V^{1/2})^{+}XQ_R$ are linear combinations of the columns of $(V^{1/2})^{+}X$ and the ranks of the two matrices are equal, there is a non-singular matrix S such that $(V^{1/2})^{+}XQ_R = (V^{1/2})^{+}X\,S$. Thus, we can write

$$X'V^{+}X\ (Q_RX'V^{+}XQ_R)^{+}X'V^{+}X = (S')^{-1}S'X'(V^{1/2})^{+}(V^{1/2})^{+}X\,[Q_R X'V^{+}XQ_R]^{+}.\ X'(V^{1/2})^{+}(V^{1/2})^{+}X\,S.S.^{-1},$$

$$= (S')^{-1}Q_R X'V^+ X Q_R [Q_R X'V^+ X Q_R]^+ Q_R X'V^+ X Q_R S^{-1}$$
$$= (S')^{-1}Q_R X'V^+ X Q_R S^{-1},$$
$$= (S')^{-1}Q_R X'(V^{1/2})^+(V^{1/2})^+ X Q_R S^{-1},$$
$$= X'(V^{1/2})^+(V^{1/2})^+ X,$$
$$= X'V^+ X,$$

and hence the theorem is proved.

As an example of the use of the theorem 5.1 in applications, consider a complete randomized block design with two treatments and two blocks:

$$\begin{bmatrix} w_{11} \\ w_{12} \\ w_{21} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{bmatrix}$$

or in matrix notations.

$$W = N\beta + e \ , \ e \sim N(0 \ , \ \sigma^2 I). \qquad (5.1)$$

To incorporate the standard assumptions $\Sigma\, \alpha_i = \Sigma\beta_j = 0$, put

$$R = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}, \ r = \begin{vmatrix} 0 \\ 0 \end{vmatrix} .$$

Premultiplication of (5.1) by

$$M = \begin{bmatrix} I_4 \\ j'_4 \end{bmatrix}$$

we get the following new model (see also Searle (1971))

$$M\,W = M\,N\,\beta + M\,e,$$

or

$$Y = X\,\beta + u, \qquad (5.2)$$

where $X = M\,N$, $Y = M\,W$, $u = M\,e$, I_4 is a 4x4 identity matrix, j_4 a 4x1 column vector of unit elements, and $u \sim N(0, \sigma^2 V)$. Therefore, V can be written as

$$V = \begin{vmatrix} I_4 & j_4 \\ j'_4 & 4 \end{vmatrix} = M\,M'.$$

Since rank (V) = 4, it is seen that V is a singular variance covariance matrix. Since V symmetric matrix, there exists an orthogonal matrix P such that P′V P = D, where D is a diagonal matrix with the characteristic roots of V displayed on the diagonal of D . Since the characteristic roots of V are 1, 1, 1, 5, 0; then we obtain

$$P = \begin{bmatrix} 1/\sqrt{12} & -2/\sqrt{6} & 0 & 1/\sqrt{20} & 1/\sqrt{5} \\ 1/\sqrt{12} & 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{20} & 1/\sqrt{5} \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{20} & 1/\sqrt{5} \\ -3/\sqrt{12} & 0 & 0 & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & 0 & 4/\sqrt{20} & -1/\sqrt{5} \end{bmatrix}.$$

Hence, from section 4, we get

$$(V^{1/2})^{+} = \left[\begin{array}{c:c} I_4 + \left(\frac{1}{10\sqrt{20}} - \frac{1}{4}\right)J_4 & \frac{4}{10\sqrt{20}}\, j_4 \\ \hdashline \frac{4}{10\sqrt{20}}\, j'_4 & \frac{16}{10\sqrt{20}} \end{array}\right]$$

and

$$V^{+} = \left[\begin{array}{c:c} I_4 - \frac{24}{100}\, J_4 & \frac{4}{100}\, j_4 \\ \hdashline \frac{4}{100}\, j'_4 & \frac{16}{100} \end{array}\right]$$

where J_4 is a 4x4 matrix of unit elements. The Moore–Penrose g–inverse of R is

$$R^{+} = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}.$$

As seen in the theorem 5.1, rank $\{(V^{1/2})^{+}XQ_R\}$ = rank $\{(V^{1/2})^{+}X\} = 3$. Furtheromore,

$$\begin{aligned} Q_R\, X'V^{+}X\, Q_R &= Q_R\,(MN)'V^{+}(MN)\,Q_R \\ &= Q_R N'M'(MM')^{+}M\,N\,Q_R \\ &= Q_R N'M'(M')^{+}M^{+}M\,N\,Q_R \\ &= Q_R N'(M^{+}M)'M^{+}M\,N\,Q_R \\ &= Q_R N'N\,Q_R, \end{aligned}$$

or clearly

$$Q_R X'V^{+}X\, Q_R = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

where $M'V^{+}M = I_4$ and $M^{+}M = I_4$.

Thus

$$C_2 = (Q_R X'V^{+}X\, Q_R)^{+} = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & -1/4 & 0 & 0 \\ 0 & -1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & 0 & -1/4 & 1/4 \end{bmatrix} = (X'V^{+}X)^{+}.$$

On the other hand, since

$$\begin{aligned} X'(I - V\,V^{+}) &= N'M'(I - MM'(MM')^{+}) \\ &= N'(M' - M'MM'(M')^{+}M^{+}) \\ &= N'(M'-M') = 0, \end{aligned}$$

we say that, X is contained in the column space of V.

If X is contained in the column space of V, the BLUE of $p_1'\beta$ is given by $p'_1\bar{\beta}$, where $p'_1 = \delta'(V^{1/2})^{+}X + \rho'R$ and $\bar{\beta}$ is given in the following theorem.

THEOREM 5.2 (Hallum, Lewis and Boullion (1973)) The best linear estimate of β in (1.1) subject to (1.2) with X,R , and V assumed to be arbitrary ranks, is given by $\bar{\beta}$, where

$$\bar{\beta} = R^{+}r + (A_1 + A_2)(Y - X\,R^{+}r)$$

with

$$A_1 = [\,(Q_R\ X'V^+X\ Q_R)\,]^+X'V^+,$$

$$A_2 = [\,Q_R\ X'(\,I - V\ V^+)\ X\ Q_R\,]^+X'(I - V\ V^+).$$

COROLLARY 5.2 In the model $Y = X\beta + u$, $u \sim N\,(0;\ \sigma^2 I)$, wherein the only restrictions on β are equations $R\beta = r$, the BLUE of a parametric function $p'_2\tilde{\beta} = p'_2R^+r + p'_2(X\,Q_R)^+\,(Y - X\,R^+r)$, where $p'_2 = \delta'X + \rho'R$. This result is fairly well known, see for example Gerig and Gallant (1975) and Baksalary and Kala (1979).

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ÖZET

Bu çalışmada lineer kısıtlamalar altında parametrelerin kestirimi incelenmiştir. R, X ve V üzerinde rank kısıtlaması yapılmamıştır. Elde edilen bulgulara dayanarak parametreler üzerindeki kestirilebilir olmayan kısıtlamalar altında normal denklemlerin çözümünü veren $X'V^+X$ nin genelleştirilmiş tersi bulunmuştur.

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