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The Space of Sequences of Which A-Transforms Are In l_p

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The Space of Sequences of Which A-Transforms Are In l_p

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SUMMARY

In this paper, we have defined and investigated two new sequence spaces $l_p(A)$ and $l_\infty(A)$ which contain the sequence spaces $X_{a(p)}$ and $X_{a(\infty)}$, given by Wang, [5], as special cases, respectively. Some inclusion theorems between the related sequence spaces have also been proved.

1. Introduction.

In [5], Wang has defined Nörlund sequence spaces $X_{a(p)}$ and $X_{a(\infty)}$ as

$$X_{a(p)} = \left\{ x = (x_k): \left(\sum_{n=0}^{\infty} \left| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \right|^p \right)^{1/p} < \infty, 1 \leq p \leq \infty \right\}$$

and

$$X_{a(\infty)} = \left\{ x = (x_k) : \sup_n \left| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \right| < \infty, n \geq 0 \right\}$$

respectively, where $a = (a_n)$ is a sequence of positive real numbers and $A_n = \sum_{k=0}^{n} a_k$. In other words, the spaces $X_{a(p)}$ and $X_{a(\infty)}$ consist of the sequences of which Nörlund transforms are in l_p and in l_∞, respectively, where

$$l_p = \{x = (x_k): \sum_{k=1}^{\infty} |x_k|^p < \infty, 1 \leq p < \infty \}$$

and

$$l_\infty = \{ x = (x_k): \sup_k |x_k| < \infty \}.$$

It is known that the sequence space

$$X_p = \{ x = (x_k) : (\sum_{n=1}^{\infty} | \frac{1}{n} \sum_{k=1}^{n} x_k |^p)^{1/p} < \infty, 1 \leq p < \infty \}$$

which has been introduced by Ng, [2], is a special case of $X_{a(p)}$ corresponding to $a_n = 1$ for every n, [5]. Obviously, the space X_p consists of the sequences of which (C,1)-transforms are in l_p.

We note that if $A = (a_{nk})$ is an infinite matrix of numbers a_{nk} $(n,k = 1,2, \ldots)$ then the sequence $(A_n(x))$ given by the matrix multiplication

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

(assuming that $A_n(x)$ exists for every $n = 1,2, \ldots$) is called the A–transform of the sequence $x = (x_k)$.

In this article, we will first define two new sequence spaces $l_p(A)$ and $l_\infty(A)$ which consist of the sequences of which any A–transform is in l_p and in l_∞, respectively. And then, we will investigate some properties of these spaces.

Throughout the paper, the triangular inequality, Minkowski's inequality, Hölder's inequality and the following inequality ([1],p.4)

$$(1) \qquad (\sum_{k=1}^{n} |a_k|^s)^{1/s} \leq (\sum_{k=1}^{n} |a_k|^r)^{1/r}, \quad (0 < r < s)$$

will be used frequently.

2. Definitions.

Let $x = (x_k)$ be a sequence of real numbers and $A = (a_{nk})$ be an infinite matrix of positive real numbers. Now, let us define

$$(2) \qquad l_p(A) = \{ x = (x_k) : \sum_{n=1}^{\infty} | \sum_{k=1}^{\infty} a_{nk} x_k |^p < \infty, 1 \leq p < \infty \}$$

and

$$(3) \qquad l_\infty(A) = \{ x = (x_k) : \sup_n | \sum_{k=1}^{\infty} a_{nk} x_k | < \infty, \ n \geq 1 \}.$$

It can easily be seen that when we take the identity matrix, (C,1) matrix and N_a matrix for $A = (a_{nk})$ in $l_p(A)$, we get the sequence spaces l_p, X_p and $X_{a(p)}$ as the special cases, respectively.

After some routine calculations it becomes clear that the spaces $l_p(A)$ and $l_\infty(A)$ are Banach spaces with the norms

$$\|x\|_p = \Big(\sum_{n=1}^{\infty} \Big|\sum_{k=1}^{\infty} a_{nk}\Big|^p\Big)^{1/p}, \ (1 \leq p < \infty)$$

and

$$\|x\|_\infty = \sup_n \Big|\sum_{k=1}^{\infty} a_{nk}x_k\Big|, \ (n \geq 1)$$

respectively.

3. Inclusion Theorems

In this paragraph, we are going to give some inclusion theorems between the related sequence spaces.

T h e o r e m 1. If $1 \leq r < s$ then $l_r(A) \subseteq l_s(A)$.

P r o o f. The proof of this theorem is an immediate consequence of the inequality (1).

T h e o r e m 2. Let $A = (a_{nk})$ be a triangular matrix of positive numbers and $x = (x_k)$ be a sequence of real numbers. If

$$a_{nn} = O(n^{-1}) \tag{4}$$

and $(a_{nk})_{1 \leq k \leq n}$ forms a non-decreasing sequence for each n then

$$ces_p \subseteq l_p(A)$$

where ces_p is the sequence space given by

$$ces_p = \Big\{ x = (x_k) : \Big(\sum_{n=1}^{\infty} \Big(\frac{1}{n}\sum_{k=1}^{n} |x_k|\Big)^p\Big)^{1/p} < \infty, 1 < p < \infty \Big\}$$

([4]).

P r o o f. Let $x = (x_k) \in ces_p$. Using the triangular inequality and condition (4), we write

$$\Big|\sum_{k=1}^{n} a_{nk}\, x_k\Big|^p \leq \Big(\sum_{k=1}^{n} a_{nk}|x_k|\Big)^p$$

$$\leq (a_{nn} \sum_{k=1}^{n} |x_k|)^p$$

$$= O(1) \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^p$$

Then, taking sum over n from 1 to ∞, we get

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} a_{nk}x_k \right|^p \leq O(1) \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^p < \infty$$

which completes the proof.

T h e o r e m 3. Let $A = (a_{nk})$ and $x = (x_k)$ be as in Theorem 2. If

(5) $$a_{n1} = O(n^{-1})$$

and $(a_{nk})_{1 \leq k \leq n}$ forms a non-increasing sequence for each n, then

$$ces_p \subseteq l_p(A).$$

P r o o f. Let $x = (x_k) \in ces_p$. Then using the triangular inequality and condition (5), we write

$$\left| \sum_{k=1}^{n} a_{nk}x_k \right|^p \leq \left(\sum_{k=1}^{n} a_{nk} |x_k|\right)^p$$

$$\leq (a_{n1} \sum_{k=1}^{n} |x_k|)^p$$

$$= O(1) \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^p$$

and so we get

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} a_{nk}x_k \right|^p \leq O(1) \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^p < \infty$$

which proves the theorem.

T h e o r e m 4. Let $A = (a_{nk})$ and $B = (b_{nk})$ be any two normal infinite matrices of positive real numbers, ([3], pp. 14–15). And let $C = (c_{nk})$ be a matrix defined by $C = BA^{-1}$ satisfying the condition

$$(6) \qquad \sum_{k=1}^{n} |c_{nk}| = O(n^{-1})$$

where A^{-1} is the inverse matrix of A, then

$$l_p(A) \subseteq l_p(B), \quad p > 1 .$$

P r o o f. Let $x = (x_k) \in l_p(A)$. Since

$$b_{nk} = \sum_{i=k}^{n} c_{ni} a_{ik}$$

by the definition of the matrix C, we write

$$\sum_{k=1}^{n} b_{nk} x_k = \sum_{k=1}^{n} \left(\sum_{i=k}^{n} c_{ni} a_{ik} \right) x_k$$

$$= \sum_{k=1}^{n} c_{nk} \sum_{i=1}^{k} a_{ki} x_i$$

Put

$$M = \sum_{k=1}^{\infty} \left| \sum_{i=1}^{k} a_{ki} x_i \right|^p .$$

Then using Hölder's inequality and condition (6), we have

$$\left| \sum_{k=1}^{n} b_{nk} x_k \right| = \left| \sum_{k=1}^{n} c_{nk} \sum_{i=1}^{k} a_{ki} x_i \right|$$

$$\leq \left(\sum_{k=1}^{n} |c_{nk}|^q \right)^{1/q} \left(\sum_{k=1}^{n} \left| \sum_{i=1}^{k} a_{ki} x_i \right|^p \right)^{1/p}$$

$$\leq M^{1/p} \sum_{k=1}^{n} |c_{nk}| ,$$

where $p^{-1} + q^{-1} = 1$. Therefore, we finally get

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} b_{nk} x_k \right|^p \leq O(1) \; M \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$$

which completes the proof.

ÖZET

Bu çalışmada, Wang tarafından[5] de verilen $X_{a(p)}$ ve $X_{a(\infty)}$ dizi uzaylarını özel hal olarak içeren $l_p(A)$ ve $l_\infty(A)$ dizi uzayları tanımlanarak,bu uzayların temel özellik leri incelenmiş ve ilgili dizi uzayları arasındaki bazı içerme bağıntıları ispatlanmıştır.

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