COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 23 A ANNÉE 1974

On |C,1| Summability Factors of Fourier Series

by

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Ankara, Turquie

Communications de la Faculté des Sciences
de l'Université d'Ankara

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On |C,1| Summability Factors of Fourier Series

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Let f (t) be a periodic function with period 2 π and integrable in the sense of Lebesgue over (– π, π), then its Fourier series is

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv$$

$$\equiv A_n (t)$$

Concerning the | C,1 | summability of Fourier series Fu Cheng Hsiang (Pac. J. of Maths. Vol. 33, No. 1, 1970) has prove dsome therems.

Our aim is to prove the same theorems under a weaker condition. Our main theorem is as follows:

Theorem. If

$$\int_{\varkappa}^{\pi} \frac{| \varphi (u) |}{u} du = 0 \left(\log \frac{1}{\varkappa}\right)$$

as t → + 0, then the series

$$\Sigma \frac{An (xo)}{n^{\alpha}}$$

is | C, 1 | summable, $\alpha > 0$.

1. Let f (t) be a periodic function with period 2π and integrable in the sense of Lebesgue over ($-\pi$, π), then its Fourier series is

$$(1.1) \quad f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

and the conjugate series

$$\Sigma\ (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n\ (t)$$

Suppose s_n denotes the partial sum of the infinite series Σa_n and let S_n^α and t_n^α denote the nth Cesaro mean of order α, $(\alpha > -1)$ of the sequences $\{s_n\}$ and $\{n\ a_n\}$ respectively. The infinite series Σa_n is said to be summable $|\ C, \alpha\ |$ if the sequence $\{s_n{}^\alpha\} \varepsilon$ BV, i.e.,

(1.2) $$\sum_{n=1}^{\infty} |\ s_n^\alpha - s^\alpha_{n-1}\ | < \infty$$

We know that

(1.3) $$t_n^\alpha = n\ (s_n^\alpha - s_{n-1}^\alpha\ )$$

so

(1.4) $$\Sigma\ |T_n^\alpha|/n < \infty$$

We shall use throughout this paper the following notations:

For a fixed point x_0, we write

$$\varphi\ (t) = \varphi x_o\ (t) = f\ (x_o + t) + f\ (x_o - t) - 2f\ (x_o)$$

$$\Phi\ (t) = \int_o^t |\ \varphi\ (u)\ |\ du;$$

and

$$\Psi\ (t) = \int_o^t |\ \psi\ (u)\ |\ du \equiv \int_o^t |\ f\ (x_0 + u) - f\ (x_0 - u)\ |\ du$$

and moreover

$\log^k n = \log\ (\log^{k-1} n)$ and $\log^2 n = \log\ (\log n)$.

2. Generalizing the previous result of Chow [1], Hsiang [2] quite recently proved the following theorems:

Theorem A. *If*

(2.1) $$\Phi\ (t) = \int_o^t |\ \varphi\ (u)\ |\ du = 0\ (t)$$

as $t \longrightarrow +0$, *then the series*

$$\Sigma\ A_n\ (x_0)\ /\ n^\alpha$$

is summable $|\ C, 1\ |$ *for every* $\alpha > 0$.

Theorem B. *If*

$$\Phi(t) = 0\left(\frac{t}{\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}}\right) \tag{2.2}$$

as t $\longrightarrow +$ o, *then the series*

$$\sum_{n=n_o}^{\infty} \frac{A_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log n)^{1+\epsilon}}, \quad (\log^k n_o > 0)$$

is summable | C,1 | *for every* $\varepsilon > 0$.

Theorem C. *If*

$$\Psi(t) = \int_o^t |\psi(u)| \, du = 0(t) \tag{2.3}$$

as t $\longrightarrow +$ o, *then the series*

$$\sum_{n=1}^{\infty} \frac{B_n(x_0)}{n^{\alpha}}$$

is summable | C,1 | *for every* $\alpha > 0$,

Theorem D. *If*

$$\Psi(t) = \int_o^t |\psi(u)| \, du = 0\left(\frac{t}{\left(\prod_{\mu=1}^{k} \log^{\mu} \frac{1}{t}\right)}\right) \tag{2.4}$$

as t $\longrightarrow +$ o, *then the series*

$$\sum_{n=n_o}^{\infty} \frac{B_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\epsilon}}, \quad (\log^k n_o > 0)$$

is summable | C,1 | *for every* $\varepsilon >$ o.

It is easily seen [3] that, if (2.1) holds then

$$\int_t^{\pi} \frac{|\varphi(u)|}{u} \, du = 0\left(\log \frac{1}{t}\right) \text{ as } t \longrightarrow o \tag{2.5}$$

but it is not true conversely.

On the other hand, if (2.5) is true, then

$$\int_0^t | \varphi (u) | \, du = 0 \left(t \log \frac{1}{t} \right) \tag{2.6}$$

and this result is best possible. Thus (2.6) is a weaker assertion than (2.1).

The object of this paper is to replace condition (2.1) by (2.5) which is a weaker one, in all the above mentioned theorems. Hence we prove the following:

Theorem 1. *If*

$$\int_t^{\pi} \frac{| \varphi (u) |}{u} \, du = 0 \left(\log \frac{1}{t} \right) \tag{2.7}$$

as t $\longrightarrow + o$, *then the series*

$\Sigma \dfrac{A_n (x_0)}{n^{\alpha}}$ *is* | C,1 | *summable. for every* $\alpha > o$.

Theorem 2. *If*

$$\int_t^{\pi} \frac{| \varphi (u) |}{u} \, du = 0 \left(\frac{\log \frac{1}{\varkappa}}{\prod\limits_{\mu=1}^{k} \log^{\mu} \frac{1}{t}} \right) \tag{2.8}$$

as t $\longrightarrow + o$, *then the series*

$$\sum_{n=n_0}^{\infty} \frac{A_n (x_0)}{\left(\prod\limits_{\mu=1}^{k-1} \log^{\mu} n \right) (\log^k n)^{1+\epsilon}}, \quad (\log^k n_0 > o)$$

is | C,1 | *summable, for every* $\varepsilon > 0$.

Theorem 3. *If*

$$\int_t^{\pi} \frac{| \psi (u) |}{u} \, du = 0 \left(\log \frac{1}{t} \right) \tag{2.9}$$

as t ⟶ + 0, *then the series*

$$\Sigma \frac{A_n(x_0)}{n^{\alpha}}$$

is | C,1 | *summable for every* $\alpha > 0$.

Theorem 4. *If*

$$(2.10) \quad \int_t^{\pi} \frac{|\psi(u)|}{u} du = 0 \left(\frac{\left(\log \frac{1}{t}\right)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\log^k n)^{1+\epsilon}} \right)$$

as t ⟶ + 0, *then the series*

$$\sum_{n=n_0}^{\infty} \frac{B_n(x_0)}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right)(\mathrm{lon}^k\, n)^{1+\epsilon}}, \quad (\log^k n_0 > 0)$$

is summable | C,1 | *for every* $\varepsilon > 0$.

3. For the proofs of these theorems, we require the following lemmas:

Lemma 1 [2]. *Let*

$$S\upsilon(t) = \sum_{\nu=o}^{n} \upsilon \cos \upsilon t$$

then

$$S\upsilon(t) = \begin{cases} 0(n^2) & (\text{for all } t) \\ 0(n/t) & (nt \geq 1) \end{cases}$$

Lemma 2. [2].

$$\left| \frac{1}{n+1} \left\{ \sum_{\nu=1}^{n} S\upsilon(t) \,\triangle\, \frac{1}{(\upsilon+2)^{\alpha}} \right\} \right| \leq \begin{cases} \frac{A^*}{tn^{\alpha}} + \frac{A}{nt^{1-\alpha}} & (t > + > 1) \\ An^{1-\alpha} & (\text{for all } t) \end{cases}$$

*A is a finite constant but is nat necevanily the same at each occuvvence.

4. *Proof of Theorem* 1

We have

$$A_n\ (x_0) = \frac{2}{\pi} \int_o^{\alpha} \varphi\ (t)\ \text{cosnt dt}.$$

Let $T_n\ (x_0)$ be the nth Cesaro mean of the first order of the sequence $\{ nA_n\ (x_0)\ /\ n^{\alpha} \}$, then

$$\frac{\pi}{2}\ T_n\ (x_0) = \int_o^{\pi} \varphi\ (t)\ \frac{1}{n+1} \sum_{\nu=o}^{n} \frac{(\upsilon+2)\cos(\upsilon+2)t}{(\upsilon+2)^{\alpha}}\ dt.$$

Abel's transformation gives

$$\frac{\pi}{2}\ T_n\ (x_0) = \int_o^{\pi} \varphi\ (t)\ \frac{1}{n+1} \left\{ \sum_{\nu=o}^{n} S\upsilon\ (t)\ \triangle\ \frac{1}{(\upsilon+2)^{\alpha}} \right\} dt$$

$$+ \int_o^{\pi} \varphi\ (t) \frac{1}{n+1}\ \frac{S_n\ (t)}{(n+3)^{\alpha}}\ dt.$$

$$= I_{1n} + I_{2n},$$

say. Thus, on writing

$$I_{1n} = \int_o^{1/n} + \int_{1/n}^{\pi} = I_{3n} + I_{4n},$$

say, we see that

$$I_{3n} = 0\ (n^{1-\alpha} \int_o^{1/n} |\varphi(t)|\ dt) = 0\ (n^{-\alpha}.\ \log n)$$

by conditions (2.6) and (2.7).

$$I_{4n} = 0 \left\{ \frac{1}{n^{\alpha}} \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\ dt \right\} + 0 \left\{ \frac{1}{n} \int_n^{\pi} \frac{|\varphi(t)|}{2-\alpha}\ dt \right\}$$

$$= 0 \left\{ \frac{1}{n^{\alpha}}\ .\ (\log n) \right\} + 0 \left\{ \frac{1}{n}\ .\ n^{1-\alpha}\ .\ (\log n) \right\}$$

since $\int_{1/n}^{\pi} \frac{|\varphi(t)|}{2-\alpha}\ dt \leq n^{1-\alpha} \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\ dt = 0\ (n^{1-\alpha}.\ \log n)$

Hence $I_{4n} = 0\ \{ n^{-\alpha}\ (\log n)\}$

by condition (2.7) of the theorem.

Now, as before, we write

$$I_{2n} = \int_{o}^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n},$$

say. Then,

$$I_{5n} = (n^{1-\alpha} \int_{o}^{1/n} |\varphi(t)| \, dt) = O\,(n^{-\alpha} \,.\, \log n)$$

and

$$I_{6n} = 0 \left\{ n^{-\alpha} \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt \right\} = O\,(n^{-\alpha} \log n),$$

by the similar arguments as in the estimation of I_{3n} and I_{4n}. But we have to show the convergence of $\Sigma\, T_n\,(x_0)\ /n$. And from the above analysis, it concludes that

$$\sum_{n=1}^{\infty} \frac{|T_n\,(x_0)\,|}{n} \leq \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \{\,|I_{3n}| + |I_{3n}| + |I_{5n}| + |I_{6n}|\}$$

$$= 0 \left\{ \sum_{n=1}^{\infty} \frac{\log n}{n^{1+\alpha}} \right\} = 0\ (1).$$

This proves theorem 1.

Proof of Theorem 2.

Let $T_n\,(x_0)$ be the nth Cesaro mean of first order of the sequence

$$\left\{ n\ A_n\ (x_0)\ /\ \left(\prod_{\mu=1}^{k-1} \log^{\mu} n \right) (\log^{k} n)^{1+\epsilon} \right\},\ (\varepsilon > o),$$

where k is a positive integer. Abel's transformation gives

$$\frac{\pi}{2} T_n\,(x_o) = \int_{o}^{\pi} \varphi\,(t) \frac{1}{n+1} \left\{ \sum_{\nu=1}^{n} S\nu\,(t) \right. \frac{1}{\triangle \left\{ \prod_{n=1}^{k-1} \log^{\mu}\,(\nu+2) \right\} \{\log^{k}(\nu+2)\}^{1+\epsilon}} dt$$

$$+ \int_0^{\pi} \varphi(t) \frac{1}{n+1} \left\{ \prod_{\mu=1}^{k-1} \frac{S_n(t)}{\log^{\mu}(n+3)\} \{\log^k (n+3)\}^{1+\epsilon}} \right.$$

$$= I_{1n} + I_{2n},$$

say, As before, we write

$$I_{1n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{3n} + I_{4n},$$

say, and

$$I_{2n} = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{5n} + I_{6n},$$

say. Since, for $\upsilon \geq n_0$,

$$\left| \triangle \frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{u}\upsilon\right) (\log^{\mu}\upsilon)^{1+\epsilon}} \right| \leq \frac{A}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} \upsilon\right) (\log^{k}\upsilon)^{1+\epsilon}},$$

we obtain

$$\left| \frac{1}{n+1} \sum_{\nu=0}^{n} S_{\upsilon}(t) \triangle \frac{1}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} (\upsilon+2) (\log^{k} (\upsilon+2) \right)^{1+\epsilon}} \right| \leq$$

$$\leq \begin{cases} \dfrac{A}{t\left(\prod_{\mu=o}^{k-1} \log^{\mu} n\right) (\log^{k} n)^{1+\epsilon}} + \\ + \dfrac{A}{t^2 \left(\prod_{\mu=1}^{k-1} \log^{\mu} \dfrac{1}{t}\right)\left(\log^{k} \dfrac{1}{t}\right)^{1+\epsilon}} & (nt \geq 1) \\ \dfrac{An}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n\right) (\log^{k} n)^{1+\epsilon}} & (\text{for all } t) \end{cases}$$

Now, by the conditions (2.6) and (2.8), we have

$$I_{3n} = 0\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^{k} n)^{1+\epsilon}}\int_{0}^{1/n}|\varphi(t)|\,dt.\right\}$$

$$= 0\left\{\frac{\log n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^{k} n)^{1+\epsilon}}\right\}.$$

$$I_{4n} = 0\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^{k} n)^{1+\epsilon}}\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}\,dt\right\}$$

$$= 0\left\{\frac{\log n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^{k} n)^{1+\epsilon}\left(\prod_{\mu=1}^{k}\log^{\mu} n\right)}\right\}$$

Finally

$$I_{5n} = 0\left\{\frac{n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^{k} n)^{1+\epsilon}}\int_{0}^{1/n}|\varphi(t)|\,dt\right\}.$$

$$= 0\left\{\frac{\log n}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)(\log^{k} n)^{1+\epsilon}}\right\}$$

$$I_{6n} = 0\left\{\frac{1}{\left(\prod_{\mu=1}^{k-1}\log^{\mu} n\right)\mathrm{lo}(g^{k} n)^{1+\epsilon}}\int_{1/n}^{\pi}\frac{|\varphi(t)|}{t}\,dt\right\}$$

$$= 0 \left\{ \frac{\log n}{\left(\prod_{\mu=1}^{k-1} \log^{\mu} n \right) (\log^{k} n)^{2+\epsilon} \left(\prod_{\mu=1}^{k} \log^{u} n \right)} \right\}$$

Thus

$$\sum_{n=1}^{\infty} \frac{|T_n (x_0) \ |}{n} = 0 \left\{ \sum_{n=n_0}^{\infty} \frac{\log n}{n \left(\prod_{\mu=1}^{k-1} \log^{\mu} n \right) (\log^{k} n)^{2+\epsilon}} \right\}$$

$$= 0 \ (1)$$

Hence the proof of Theorem 2 is complete.

The proofs of Theorems 3 and 4 can be given on the same lines.

The authar iî thankful to Mr. D. C. Maheshwari of Muslim University Aligarh for going through the manuscript of the paper.

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Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara – 1976