

COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A₁: Mathématiques

TOME 29

ANNÉE 1980

Invariant Means And Multiplicative Matrices

by

Z.U. AHMAD, S.K. SARASWAT and MURSALEN.

7

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

**Communications de la Faculté des Sciences
de l'Université d'Ankara**

Comité de Redaction de la Série A₁

H. Hacısalihođlu, M. Oruđ, B. Yurtsever

Secrétaire de Publication

Ö. Çakar

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les diciplines scientifique représentées à la Faculté des Sciences de l'Université d'Ankara.

La Revue, jusqu'à 1975 à l'exception des tomes I, II, III etait composé de trois séries

- Série A: Mathématiques, Physique et Astronomie,
- Série B: Chimie,
- Série C: Sciences Naturelles.

A partir de 1975 la Revue comprend sept séries:

- Série A₁: Mathématiques,
- Série A₂: Physique,
- Série A₃: Astronomie,
- Série B: Chimie,
- Série C₁: Géologie,
- Série C₂: Botanique,
- Série C₃: Zoologie.

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté des Sciences de l'Université d'Ankara. Elle accepte cependant, dans la mesure de la place disponible les communications des auteurs étrangers. Les langues Allemande, Anglaise et Française seront acceptées indifféremment. Tout article doit être accompagnés d'un resume.

Les articles soumis pour publications doivent être remis en trois exemplaires dactylographiés et ne pas dépasser 25 pages des Communications, les dessins et figures portés sur les feuilles séparées devant pouvoir être reproduits sans modifications.

Les auteurs reçoivent 25 extraits sans couverture.

l'Adresse : Dergi Yayın Sekreteri,
Ankara Üniversitesi,
Fen Fakültesi,
Beşevler-Ankara

Invariant Means And Multiplicative Matrices

Z.U. AHMAD, S.K. SARASWAT* and MURSALEEN,

(Received on 18 November, 1980 and accepted on 16, June 1980)

ABSTRACT

Schaefer [7] has introduced the concept of σ conservative and σ regular matrices and obtains necessary and sufficient conditions to characterize these classes of matrices. In the present paper authors have defined $(c, V_\sigma)\beta$ -matrices and $(\omega_p, V_\sigma)\beta$ and obtain necessary and sufficient conditions to characterize them.

1. INTRODUCTION.

Let l_∞ , c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = \{x_k\}$ with usual norm $\|x\| = \sup_k |x_k|$. Let σ be a mapping of the set of positive integers into itself. A continuous linear function ϕ on l_∞ is said to be an invariant mean or a σ -mean if and only if (i) $\phi(x) \geq 0$ for all n , (ii) $\phi(e) = 1$, where $e = \{1, 1, 1, \dots\}$, and (iii) $\phi\{x_{\sigma(n)}\} = \phi(x)$ for all $x \in l_\infty$. Throughout this paper we deal only with mappings which are one to one such that $\sigma^m(n) \neq n$ for all n and m where $\sigma^m(n)$ denotes the m^{th} iterate of the mapping at n .

For such mappings, every σ -mean extends the limit functional on c . (see Raimi [6]). Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ -means are equal. When $\sigma(n) = n+1$, V_σ is the set of almost convergent sequences (see Lorentz [1]).

P. Schaefer [7] has defined the concept of σ -conservative and σ -regular and characterized these classes of matrices i.e. (c, V_σ) , $(c, V_\sigma)_{reg}$. Recently Mursaleen [5] characterized the classes of (w_p, V_σ) and $(w_p, V_\sigma)_{reg}$. Eizen and Laush [1] obtained necessary and sufficient conditions to characterize the

*Correspondence should be made on the address:

S.K. SARAWAT, Department of Mathematics, Aligarh Muslim University, ALIGARH-202001 (INDIA)

matrices of the class $(c, f)_\beta$. In the sequel, the object of this paper is to obtain necessary and sufficient conditions to characterize the matrices of the classes of $(c, V_\sigma)_\beta$ and $(w_p, V_\sigma)_\beta$, which will fill up a gap in the existing literature.

2. Preliminaries:

If p_k is real such that $p_k > 0$ and $\sup_k p_k < \infty$, we define (Maddox [4])

$$w(p) = \left\{ x : \frac{1}{n} \sum_{k=1}^n |x_k - l|^{p_k} \rightarrow 0 \text{ for some } l \right\}.$$

If E is a subset of S , the space of complex sequences, then we shall write E^+ for the generalized Köthe - Toeplitz dual of E , i. e.

$$E^+ = \left\{ a : \sum_k a_k x_k \text{ converges for every } x \in E \right\}$$

If $0 < p_k \leq 1$, then $w^+(p) = M$, where

$$M = \left[a : \sum_{r=0}^{\infty} \max_r \left\{ (2^r N^{-1})^{1/p_k} |a_k| \right\} < \infty \text{ for some} \right.$$

integer $N > 1$ and \max_r is the maximum taken over $2^r \leq k$

$< 2^{r+1}$] (see Lascarides and Maddox [2]).

When $p_k = p \forall k$, we have $w(p) = w_p$.

If x is a topological linear space we shall denote x^* the continuous dual of x , i.e. the set of all continuous linear functionals on x .

If $x = \{x_n\}$, write $Tx = \{x_{\sigma(n)}\}$. It is easy to show that the set V_σ can be characterized as the set of all bounded sequences for which $\lim_m (x + Tx + \dots + T^m x) / (m + 1)$ exists in l_∞

and has the form $L e$, $L = \sigma - \lim x$.

Throughout this paper we shall use the notation $a(n, k)$ to denote the element a_{nk} of the matrix A , for $m \geq 0$, we have

$$\begin{aligned} & (Ax + TAx + \dots + T^m Ax) / (m + 1) \\ &= \left\{ \sum_k [a(n, k) + a(\sigma(n), k) + \dots + a(\sigma^m(n), k)] x_k / (m + 1) \right\}_{n=1}^{\infty} \end{aligned}$$

where $\sigma^m(n)$ denotes the m^{th} iterate of σ at n . For every $n \geq 1$,

put

$$T_{mn}(x) = t_{mn}(Ax) = \sum_{k=1}^{\infty} \sum_{j=0}^m a(\sigma^j(n), k) x_k / (m + 1) \\ = \sum_k \alpha(n, k, m) x_k$$

where

$$\alpha(n, k, m) = \frac{1}{m + 1} \sum_{j=0}^m a(\sigma^j(n), k).$$

3. $(c, V_{\sigma})_{\beta}$ - matrices:

The notion of $(c, f)_{\beta}$ - matrices (Eizen and Laush [1]) can be generalized as follows.

DEFINITION 1. An infinite matrix A is said to be $(c, V_{\sigma})_{\beta}$ -matrix if and only if it is σ - conservative and $\sigma - \lim Ax = \lim \beta x$ for all $x \in c$.

THEOREM 1. $A \in (c, V_{\sigma})_{\beta}$ if and only if

$$\|A\| = \sup_n \left\{ \sum_k |a_{nk}| \right\} < + \infty \tag{3.1}$$

$$a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in V_{\sigma} \text{ with } \sigma\text{-limit zero for each } k \tag{3.2}$$

$$a = \left\{ \sum a_{nk} \right\}_{n=1}^{\infty} \in V_{\sigma} \text{ with } \sigma\text{-limit } \beta. \tag{3.3}$$

PROOF: Suppose that $A \in (c, V_{\sigma})_{\beta}$. If $x \in c_0$, then $Ax \in V_{\sigma} \subset I_{\infty}$. It follows from the proof of Theorem 1 [7] that $\|A\| < + \infty$. Define $e = \{1, 1, 1, \dots\}$ and $e^k = \{0, 0, 0, 1 \text{ (k}^{\text{th}}\text{-place), } 0, 0\}$. Since $Ae = a$ and $Ae^k = a_{(k)}$, (3.2) and (3.3) must hold i.e., $\sigma\text{-lim } Ae^k = 0 = \sigma - \lim a_{(k)}$ and $\sigma\text{-lim } Ae = \beta = \sigma\text{-lim } a$.

Conversely, let us suppose that (3.1), (3.2) and (3.3) must hold. Let $x \in c$, we have from the proof of Theorem 1 [7] that

$$\|T_{mn}\| < + \infty.$$

Therefore the hypothesis of Theorem 1 [7] holds with $u_k = 0$ and $u = \beta$ and so A is σ - conservative.

Therefore

$$\lim_m T_{mn}(x) = \lim [T_n(e) - \sum_k T_n(e^k)] x + \sum_k x_k T_n(e^k)$$

$$= \lim \beta x,$$

where

$$\lim_m T_{mn}(e) = T_n(e) = \beta \text{ and } \lim_m T_{mn}(e^k) = T_n(e^k) = 0.$$

So that $A \in (c, V_\sigma)\beta$.

4. $(w_p, V_\sigma)\beta$ - matrices.

DEFINITION 2. An infinite matrix A is said to be $(w_p, V_\sigma)\beta$ -matrix if and only if it is (w_p, V_σ) -matrices and the σ - $\lim Ax = \lim \beta x$ for all $x \in w_p$.

THEOREM 2. Let $1 \leq p < \infty$, then $A \in (w_p, V_\sigma)\beta$ if and only if

$$D(A) = \sup_m \sum_r 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} < +\infty$$

for every n , where $1/p + 1/q = 1$,

$$a_{(k)} = \{a_{nk}\}_{n=1}^\infty \in V_\sigma \text{ with the } \sigma\text{-limit zero for each } k, \quad (4.2)$$

$$a = \left\{ \sum_k a_{nk} \right\}_{n=1}^\infty \in V_\sigma \text{ with the } \sigma\text{-limit } \beta. \quad (4.3)$$

PROOF : Suppose that $A \in (\omega P, V_\sigma)\beta$. Define $e = \{1, 1, 1, \dots\}$ and $e^k = \{0, 0, 0, 1 \text{ (k}^{\text{th}} \text{ place)}, 0, 0, \dots\}$. Since $Ae = a$ and $Ae^k = a_{(k)}$, (4.2) and (4.3) must hold, i.e. $\sigma\text{-}\lim Ae^k = 0 = \sigma\text{-}\lim a_{(k)}$ and $\sigma\text{-}\lim Ae = \beta = \sigma\text{-}\lim a$. For the necessity of (4.1), suppose that $T_{m^n}(x) = \sum_r [\alpha(n, k, m) x_k]$ exists for each $x \in \omega p$ then for

each m and $r \geq 0$, define $f_{rm} = \sum_r \alpha(n, k, m) x_k$. Then $\{f_{rm}\}_m$ is a sequence of continuous linear functionals on w_p , since

$$\begin{aligned} |f_{rm}(x)| &\leq \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} \left(\sum_r |x_k|^p \right)^{1/p} \\ &\leq 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} \|x\|. \end{aligned}$$

It follows ([4], corollary on pp. 114) that for each m

$$\lim_j \sum_{r=0}^j f_{rm}(x) = T_{mn}(x)$$

is in the dual space w_p^* , whence there exists K_{mn} such that

$$|T_{mn}(x)| \leq K_{mn} \|x\|. \quad (4.4)$$

For each m we take any integer $j > 0$ and defining $x \in w_p$ as in ([4], Theorem 7), we have

$$\sum_{r=0}^j 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} \leq K_{mn}$$

whence, for each m

$$\sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} \leq K_{mn} \leq \infty. \quad (4.5)$$

Now, since $T_{mn}(x)$ is absolutely convergent, we have

$$|T_{mn}(x)| \leq \sum_{r=p}^{\infty} 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} \|x\|,$$

so that

$$K_{mn} \leq \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q}, \quad (4.6)$$

By virtue of (4.5) and (4.6),

$$K_{mn} = \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q}$$

Finally, (by Theorem 11 ([4]*), pp. 114) for every n , the existence of $\lim_m T_{mn}(x)$ on w_p implies that

$$\sup_m K_{mn} = \sup_m \sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} < \infty,$$

which is (4.1).

Conversely, let us suppose that conditions (4.1), (4.2) and (4.3) hold and $x \in w_p$. Now

$$\begin{aligned} |T_{mn}(x)| &\leq \sum_{r=0}^{\infty} \sum_r |\alpha(n, k, m) x_k| \\ &\leq \sum_{r=0}^{\infty} \left(\sum_r |\alpha(n, k, m)|^q \right)^{1/q} \left(\sum_r |x_k|^p \right)^{1/p} \\ &\leq D(A) \|x\|. \end{aligned}$$

Therefore $T_{mn}(x)$ is absolutely and uniformly convergent for each m . Also $\sum_{r=0}^{\infty} 2^{r/p} (\sum_r |u_k|^q)^{1/q} < \infty$, and by Holder's inequality $\sum_r u_k x_k < \infty$. Therefore the hypothesis of Theorem 2 holds with $u_k = 0$ and $u = \beta$ and so $A \in (w_p, V\sigma)$.

Therefore,

$$\begin{aligned} \lim_m T_{mn}(x) &= \lim [T_n(e) - \sum_k T_n(e^k)] x + \sum_k x_k T_n(e^k) \\ &= \lim \beta x. \end{aligned}$$

where $\lim_m T_{mn}(e) = T_n(e) = \beta$ and $\lim T_{mn}(e^k) = T_n(e^k) = 0$ so that $A \in (\omega_p, V\sigma)_\beta$.

5. We have the following corollaries.

It we take $\beta = 1$ in Theorem 1, we have

Corollary 5.1: (see Schaefer [7]). The matrix A is σ -regular if and only if

$$\|A\| < +\infty$$

$$a_{(k)} \in V\sigma \text{ with } \sigma\text{-limit zero each } k, \text{ and} \quad (5.1.2)$$

$$a \in V\sigma \text{ with } \sigma\text{-limit } +1 \quad (5.1.3)$$

If we take $\beta = 1$ in Theorem 2, we get

Corollary 5.2: (see Mursaleen [5].) Let $0 < p < \infty$, Then $A \in (w_p, V\sigma)_{reg}$ if and only if conditions (4.1), (4.2) with σ -limit = 0 and (4.3) with σ limit + 1.

If we put $\sigma(n) = n+1$ in Theorem 1, we have

Corollary 5.3: (see Eizen and Laush [1]). $A \in (c,f)_\beta$ - matrix with multiplier β if and only if

$$\|A\| < +\infty \quad (5.3.1)$$

$$\lim_n \alpha(n, k, m) = 0 \text{ (uniformly in } m, k, \text{ fixed)} \quad (5.3.2)$$

$$\lim_n \sum_k \alpha(n, k, m) = \beta \text{ (uniformly in } m) \quad (5.3.3)$$

REFERENCES

- 1- Eizen, C. and Laush, G., *Infinite matrices and almost convergence*; Math. Japon. 14, (1969), 137-143.
- 2- Lascarides, C. G. and Maddox, I.J., *Matrix transformations between some classes of sequences*; Proc. Camb. Phil. Soc., 68 (1970), 99-104.
- 3- Lorentz, G. G., *A contribution to the theory of divergent sequences* Acta Math. 80 (1948), 167-190.
- 4- Maddox, I.J., *Elements of functional analysis*, Camb. Univ. Press 1970.
- 5- Mursaleen, *Infinite matrices and Invariant means*, Ph. D. Thesis Aligarh Muslim University, Aligarh (India) 1979.
- 6- Raimi, R.A., *Invariant means and invariant matrix methods of summability*, Duke Math. Jour., 30 (1963), 81-94.
- 7- Schaefer, P., *Infinite matrices and invariant means*; Proc. Amer. Math. Soc., 36 (1) (1972), 104-110.

ÖZET

σ konservatif ve σ regüler matris kavramları Schaefer tarafından tamamlanmış olup bu matris sınıflarını karakterize eden gerek ve yeter koşullar elde edilmiştir. (7). Bu çalışmanın amacı ise $(c, V\sigma)$ $(\omega_p, V\sigma)_\beta$ - matrislerini tanımlamak ve bunları karakterize edecek gerek ve yeter koşulları elde etmektir.

Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi
Dekanlığı Ankara, Turquie.