COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A: Mathématiques, Physique et Astronomie

TOME 23 A ANNÉE 1974

On The Integral Modulus of Continuity of Fourier Series III

by

BABU RAM

12

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences de l'Université d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Univereité d'Ankara" est un organe de publication englobant toutes les disciplines scientifiques représentées à la Faculté: Mathématiques pures et appliquées, Astronomie, Physique et Chimie théorique, expérimentale et techinique, Géologie, Botanique et Zoologie.

La Revue, á l'exception des tomes I, II, III, comprend de trois séries

Série A: Mathématiques, Physique et Astronomie.

Série B: Chimie.

Série C: Sciences naturelles.

En principe. la Revue est réservée aux mémoires originaux des membres de la Faculté. Elle accepte cependant, dans la mesure de la place disponible, les communications des auteurs étrangers. Les langues allemande, anglaise et française sont admises indifféremment. Les articles devront être accompagnés d'un bref sommaire en langue turque.

Adres: Fen Fakültesi Tebliğler Dergisi Fen Fakültesi, Ankara, Turquie

On The Integral Modulus of Continuity of Fourier Series III

by

BABU RAM*

(Received June 12, 1974)

1. *Definitions and Notations:* Let F (x) be a function with period 2π in L_p $(1 \leq p < \infty)$. Then the L_p–modulus of continuity of order $k \geq 1$ of F is defined by

$$\omega^k_p (\delta; F) = \sup_{0<|t|\leq\delta} \| \Delta^k_t F(x) \| L_p,$$

where

$$\Delta_t^k F(x) = \sum_{\gamma=0}^{k} (-1)^{k-\gamma} \binom{k}{\gamma} F(x + \gamma t)$$

and $\|.\|$ denotes the norm.

Let f and g be even and odd integrable functions respectively with period 2π and let

$$f(x) \sim \tfrac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx,$$

$$g(x) \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

Throughout this paper the letter C with or without subscript denotes an absolute constant which may have different values in different contexts and depends on the subscripts.

2. Concerning the integral modulus of continuity of order 1, Aljančic [1] proved Delete the following theorem.

* Department of Mathematics, Kurukshetra University, Kurukshetra, India.

Theorem A. Let $\{a_n\}$ be a sequence which is monotonically decreasing to zero and such that for a fixed p $(1 < p < \infty)$

$$\sum_{n=1}^{\infty} n^{p-2} a_n^{\,p} < \infty.$$

Then

$$\omega_p (1/n;f) \leq C_p \; n^{-1} \left[\sum_{v=1}^{n-1} v^{2p-2} a^p_v \right]^{1/p}$$

$$+ \; C_p \left[\sum_{v=n}^{\infty} v^{p-2} a^p_v \right]^{1/p}.$$

The result holds for sine series also.

Later on Aljančic [2] generalized Theorem A for higher order of integral modulus of continuity in the following form:

Theorem B, If $a_n \downarrow o$ and

$$\sum_{m=1}^{\infty} m^{p-2} a^p_m < \infty \qquad (1 < p < \infty),$$

then

$$\omega_p^{\,k} (1/n;\, f) \leq C_{k,p} \left[n^{-k} \left(\sum_{m=1}^{n} m^{(k+1)p-2} a^p_m \right)^{-1/p} \right.$$

$$\left. + \left(\sum_{m=n+1}^{\infty} m^{p-2} a^p_m \right)^{1/p} \right].$$

An analogous result holds for g also.

In (1968), Izumi [6] generalized Theorem A in the sense that they replaced its conditions by a weaker condition

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{m=n}^{\infty} | \, \Delta \, a_m \, | \right)^p < \infty.$$

They established the following theorem:

Theorem C. If

$$\sum_{n=1}^{\infty} n^{p-2} \left(\sum_{m=n}^{\infty} | \Delta a_m | \right)^p < \infty \ (1 < p < \infty),$$

then

$$\omega_p (h;f)^p \leq C_p h^p \sum_{n \leq 1/h} n^{2p-2} \left(\sum_{m=n}^{\infty} | \Delta a_m | \right)^p$$

$$+ C_p \sum_{n > 1/h} n^{p-2} \left(\sum_{m=n}^{\infty} | \Delta a_m | \right)^p.$$

The result holds for g also.

3. The object of this paper is to generalize Theorem C for the integral modulus of continuity of higher order. In what follows we shall establish the following

Theorem. If

$$(3.1) \quad \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{m=n}^{\infty} | \Delta a_m | \right)^p < \infty \quad (1 < p < \infty),$$

then

$$(3.2) \quad \omega_p^k (1/n;f)^p \leq C_{k,p} n^{-kp} \sum_{v \leq n} v^{(k+1)p-2} \left(\sum_{m=v}^{\infty} | \Delta a_m | \right)^p$$

$$+ C_{k,p} \sum_{v > n}^{\infty} v^{p-2} \left(\sum_{m=1}^{\infty} | \Delta a_m | \right)^p.$$

The theorem also holds for g.

The following corollary is a consequence of our theorem.

Corollary. If

$$(3.3) \quad \sum_{m=v}^{\infty} | \Delta a_m | \leq C | a_v |, \quad \sum_{v=1}^{\infty} v^{p-2} | a_v |^p < \infty,$$

then

$$(3.4) \quad \omega_p^k (1/n;f)^p \leq C_{k,p} n^{-kp} \sum_{v \leq n} v^{(k+1p-2} | a_v |^p$$

$$+ C_{k,p} \sum_{v > n} v^{p-2} | a_v |^p.$$

Analogous results holds for g.

This corollary is a generalisation on Theorem B, since the conditions of Theorem B imply (3.3).

4. *Proof of the theorem.* Let $\{a_n\}$ be the sequence of Fourier cosine coefficients of f and a (u) be the function defined on the interval $(1, \infty)$ such that

$$a(n) = a_n \qquad (n = 1,2,\dots)$$

and a (u) is linear in each interval (n, n+1) and further a (u) is continuous in the whole interval.

The differential coefficient $a'(u)$ exists for all non-integral u and $a'(u) = -\Delta a_n$ for $n < u < n+1$ $(n=1,2,\dots)$.

Under the assumed hypothesis it follows that

$$\int_M^{N-1} |a(u)|\, du \leq \sum_{n=M}^{N} |a_n| \leq C \int_M^N |a(u)|\, du.$$

Hence the condition (3.1) reduces to

$$(4.1) \quad \int_1^\infty u^{p-2} \left(\int_u^\infty |a'(v)|\, dv \right)^p du < \infty$$

and the inequality to be proved is equivalent to

$$(4.2) \quad \omega_p^k (h;f)^p \leq C_{k,p}\, h^{kp} \int_0^{1/h} u^{(k+1)p-2} \left(\int_u^\infty |a'(v)|\, dv \right)^p du$$

$$+ C_{k,p} \int_{1/h}^\infty u^{p-2} \left(\int_u^\infty |a'(v)|\, dv \right)^p du$$

$$= L_1 + L_2,$$

say. Let $h = \pi/n$. To prove the theorem it is sufficient to show that

$$(4.3) \quad I = \int_0^\pi |\Delta^k_{\pm t} f(x)|^p\, dx \leq L_1 + L_2$$

for $t \leq \pi/n$.

We may write

$$(4.4)\quad I = \int_0^{(k+1)\pi/n} | \Delta^k_{\pm t} f(x) |^p dx$$

$$+ \int_{(k+1)\pi/n}^{\pi} | \Delta^k_{\pm t} f(x) |^p dx$$

$$= I_1 + I_2,$$

say. We shall evaluate I_1 and I_2 separately. Firstly to estimate I_1, we see that

$$(4.5)\quad I_1 \leq \int_0^{(k+1)\pi/n} | \sum_{v \leq n/[(k+1)\pi]} a_v \Delta^k_{\pm t} \cos vx |^p dx$$

$$+ \int_0^{(k+1)\pi/n} | \Delta^k_{\pm t} \sum_{n/[(k+1)\pi < v \leq 1/x} a_v \cos vx |^p dx$$

$$+ \int_0^{(k+1)\pi/n} | \Delta^k_{\pm t} \sum_{v > 1/x} a_v \cos vx |^p dx$$

$$= I_3 + I_4 + I_5,$$

say. We estimate I_3 first. Since

$$\Delta^k_{\pm t} \cos vx = (-1)^k \, \mathrm{R} \, [e^{ixv} (1-e^{\pm ivt})^k]$$

$$= (-1)^k 2^k \sin^k (vt/2) \cos [vx \pm kvt/2 + k\pi/2],$$

using the inequalities $|\sin x| \leq |x|$ and $|\cos x| \leq 1$, it follows that

$$| \sum_{v \leq n/[(k+1)\pi]} a_v \Delta^k_{\pm t} \cos vx | \leq t^k \sum_{v \leq n/[k+1)\pi]} v^k |a_v|.$$

Therefore we have

$$I_3 \leq C_{k,p} n^{-kp} \int_0^{(k+1)\pi/n} \sum_{v \leq n/[(k+1)\pi]} v^k |a_v|)^p dx$$

$$\leq C_{k,p} n^{-kp-1} \left(\int_0^{n/[(k+1)\pi]} u^k |a(u)| du \right)^p.$$

Applying Holder's inequality, we have

$$(4.6)\quad I_3 \leq C_{k,p}\, n^{-kp-1} \int_0^{n/[(k+1)\pi]} |\, a\, (u)\, |^p\, u^{(k+1)p-2}\, du$$

$$>< \left(\int_0^{n/[k+1)\pi]} u^{k-2}\, du \right)^{p/2} (1/p + 1/q = 1)$$

$$\leq C_{k,p}\, n^{-kp} \int_0^{n/[(k+1)\pi]} |\, a\, (u)\, |^p\, u^{(k+1)p-2}\, du$$

$$\leq L_1.$$

Now to estimate I_4, we have

$$(4.7)\quad I_4 \leq \sum_{\gamma=0}^{k} (\gamma^k) \int_{\pm\gamma t}^{(k+1)\pi/n \pm \gamma t} \Big| \sum_{n/[(k+1)\pi] < v \leq 1/x}$$

$$a_v \cos vx\, |^p\, dx$$

$$< 2^{k+1} \int_0^{(2k+1)\pi/n} \Big| \sum_{n/[(k<1)\pi] < v \leq 1/x} a_v \cos vx\, |^p dx$$

$$= C_k \int_0^{(2k+1)\pi/n} \sum_{n/[(k+1)\pi] < v \leq 1/x} |\, a_v\, |^p\, dx$$

$$\leq C_k \int_0^{(2k+1)\pi/n} dx \left(\int_{n/[(k+1)\pi]}^{1/x} |\, a\, (u)\, |\, du \right)^p$$

$$\leq C_k \int_0^{(2k+1)\pi/n} |\, a\, (1/x)\, |^p\, x^{-p}\, dx$$

$$\leq C_k \int_{n/[(2k+1)\pi]}^{\infty} |\, a\, (u)\, |^p\, u^{p-2}\, du$$

$$= C_k \left(\int_{n/[(2k+1)\pi]}^{n/\pi} + \int_{n/\pi}^{\infty} \right)$$

$$\leq L_1 + L_2.$$

Let $\tau = [1/x]$, that is, the integral part of $1/x$. Then to estimate I_5 we use Abel's transformation and observe that

$$(4.8)\quad I_5 \leq \sum_{\gamma=0}^{k} \binom{k}{\gamma} \Big[2 \int_0^{(2k+1)\pi/n} | a_{\tau+1} D_{\tau+1}(x) + \sum_{v>\tau+1} \Delta a_v D_v(x) |^p dx \Big]$$

$$\leq 2^{k+1} \int_0^{(2k+1)\pi/n} | a_{\tau+1} D_{\tau+1}(x) |^p dx$$

$$+ 2^{k+1} \int_0^{(2k+1)\pi/n} | \sum_{v>\tau+1} \Delta a_v D_v(x) |^p dx$$

$$= S_1 + S_2,$$

say. Then

$$(4.9)\quad S_1 = 2^{k+1} \sum_{m=n+1}^{\infty} \int_{(2k+1)\pi/m}^{(2k+1)\pi/(m-1)} | a_{\tau+1} D_{\tau+1}(x) |^p dx$$

$$\leq C_k \sum_{m=n+1}^{\infty} \int_{(2k+1)\pi/m}^{(2k+1)\pi/(m-1)} | 1/x \, a(1/x) |^p dx$$

$$= C_k \int_0^{(2k+1)\pi/n} x^{-p} | a(1/x) |^p dx$$

$$\leq C_k \int_{n/[(2k+1)\pi]}^{\infty} | a(u) |^p u^{p-2} du$$

$$= C_k \Big[\int_{n/[(2k+1)\pi]}^{n/\pi} + \int_{n/\pi}^{\infty} \Big]$$

$$\leq L_1 + L_2.$$

Moreover,

$$(4.10)\quad S_2 \leq \sum_{\gamma=0}^{k} \binom{k}{\gamma} \Big[2 \int_0^{(2k+1)\pi/n} | \sum_{v>\tau+1} \Delta a_v D_v(x) |^p dx \Big]$$

$$\leq 2^{k+1} \sum_{m=n+1}^{\infty} \int_{(2k+1)\pi/m}^{(2k+1)\pi/(m-1)} dx\, x^{-p} \left[\int_x^{\infty} |a'(u)|\, du \right]^p$$

$$= C_k \int_{n/[(2k+1)\pi]}^{\infty} x^{p-2} \left[\int_x^{\infty} |a'(u)|\, du \right]^p$$

$$= C_k \left[\int_{n/[(2k+1)\pi]}^{n/\pi} + \int_{n/\pi}^{\infty} \right]$$

$$\leq L_1 + L_2.$$

Combining (4.9) and (4. 10), we have

(4.11) $\qquad I_5 \leq L_1 + L_2.$

Recalling (4.6), (4.7) and (4.11), it follows that

(4.12) $\qquad I_1 \leq L_1 + L_2.$

It remains to estimate I_2. We write

(4.13) $$I_2 \leq \int_{(k+1)\pi/n}^{\pi} \left| \sum_{v \leq \tau} a_v \Delta^k_{\pm t} \cos vx \right|^p dx$$

$$+ \int_{(k+1)\pi/n}^{\pi} \left| \Delta^k_{\pm t} \sum_{v > \tau} a_v \cos vx \right|^p dx$$

$$= I_6 + I_7$$

say. As in the estimation of I_3, we have

(4.14) $$I_6 \leq C_k n^{-kp} \int_{(k+1)\pi/n}^{\pi} \left(\sum_{v \leq \tau} v^k |a_v| \right)^p dx$$

$$\leq C_k n^{-kp} \int_{(k+1)\pi/n}^{\pi} \left(\int_1^{1/x} u^k |a(u)|\, du \right)^p dx$$

$$\leq C_k n^{-kp} \int_{(k+1)\pi/n}^{\pi} x^{-kp} \left(\int_1^{1/x} |a(u)|\, du \right)^p du,$$

which by means of Hardy's inequality [5, Theorem 330)] is

$$\leq C_k\, n^{-kp} \int_{(k+1)\pi/n}^{\pi} x^{-(k+1)p} \mid a\,(1/x) \mid^p dx$$

$$\leq C_k\, n^{-kp} \int_{1}^{n/[(k+1)\pi]} \mid a\,(u) \mid^p u^{(k+1)p-2}\, du$$

$$\leq L_1.$$

Moreover, by virtue of Abel's transformation

$$(4.15)\quad I_6 \leq \sum_{\gamma=0}^{k} \binom{k}{\gamma} \int_{k+1)\pi/n\pm\gamma t}^{\pi\pm\gamma t} \mid \sum_{v>\tau} \Delta a_v D_v\,(x) \mid^p dx$$

$$+ \sum_{\gamma=0}^{k} \binom{k}{\gamma} \int_{(k+1)\pi/n\mp\gamma t}^{\pi+\gamma t} \mid a_{\tau+1}\, D_{\tau+1}\,(x) \mid^p dx$$

$$= S_3 + S_4,$$

say. Then we have

$$(4.16)\quad S_3 \leq C_k \int_{\pi/n}^{\infty} \sum_{v>\tau} \Delta\, a_v\, D_v\,(x) \mid^p dx$$

$$\leq C_k \int_{\pi/n}^{\infty} x^{-p} \mid \int_{1/x}^{\infty} a'\,(u)\, du \mid^p dx$$

$$\leq C_k \int_{0}^{n/\pi} v^{p-2} \mid \int_{v}^{\infty} a'\,(u)\, du \mid^p dv.$$

Since $2-p < 1$ $(1 < p < \infty)$ it follows by measn of Hardy's inequality [5, Theorem 328] that

$$(4.17)\quad S_3 \leq C_k \int_{0}^{n/\pi} v^{2p-2} \mid a'(v) \mid^p dv \leq L_1 + L_2.$$

On the other hand

$$(4.18)\quad S_4 \leq C_k \int_{\pi/n}^{\pi+k\pi/n} \mid a\,(1/x) \mid^p x^{-p}\, dx$$

$$\leq C_k \int_0^{n/\pi} | a(u) |^p u^{p-2} du \leq L_1.$$

Thus by virtue of (4.15), (4.17) and (4.18), we have

(4.19) $$I_7 \leq L_1 + L_2.$$

Combining (4.14) and (4.19), we obtain

(4.20) $$I_2 \leq L_1 + L_2.$$

Hence, the inequalities (4.12) and (4.20) imply that

$$I \leq L_1 + L_2.$$

This concludes the proof of our Theorem for f.

The proof for g is similar. So we omit it.

REFERENCES

[1] **S. Aljancic',** *On the integral moduli of continuity in Lp* $(1<p<\infty)$ *of Fourier series with monotone coefficients,* .Proc. Amer. Math. Soc. 17 (1966), 287–294.

[2] **S. Aljancic',** *Sur le module des séries de Fourier particulièrs et sur le module des series de Fourier transformees par des multiplicateurs de type divers, Bull.* Acad. Serbe Sc. Arts, classe sc. mat. nat. T. XL, No 6 (1967), 13–38.

[3] **Babu Ram.** *On the integral modulus of continuity of Fourier Series,* 1,*Bull.* del' Acad. royale de Belgique (classe des sciences), T. LVIII, Fasc. 3 (1972), 347–353.

[4] **Babu Ram,** *On the integral modulus of continuty of Fourier Series II. Jour.* D' Anal. Math. (qn the press).

[5] **G.H. Hardy. J. E. Littlewood and G. Polya,** *Inequalities,* Cambridge University Press, 1959.

[6] **M. Izumi and S. Izumi,** *Modulus of continuity of functions defined by trigonometric series, Jour.* Math. Anal. Appl., 24 (1968), 564–581.

Prix de l'abonnement annuel

Turquie : 15 TL; Étranger: 30 TL.

Prix de ce numéro : 5 TL (pour la vente en Turquie).

Prière de s'adresser pour l'abonnement à : Fen Fakültesi Dekanlığı Ankara, Turquie.

Ankara Üniversitesi Basımevi. Ankara – 1976