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Araştırma Makalesi / Research Article

The Approximate Solution of Singularly Perturbed Burger-Huxley Equation with RDTM

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Abstract
In this study, reduced differential transform method (RDTM) is proposed to solve singularly perturbed Burger-Huxley partial differential equation. Firstly, this equation is transformed to algebraic equation. Then, recurrence relation and differential transform coefficients are obtained. Finally, highly accurate approximate solutions of this equation are found for three examples.

Keywords: Singularly Perturbed Burger-Huxley Partial Differential Equation, Approximate Solution, Boundary Layer, RDTM.

RDTM ile Singüler Pertürbe Burger-Huxley Denkleminin Yaklaşık Çözümü

Öz
Bu çalışmada, singuler pertürbe Burger-Huxley kısmi türevli diferansiyel denklemini çözmek için indirgenmiş diferansiyel dönüşüm yöntemi (RDTM) önerilmiştir. İlk olarak bu denklem cebirsel denkleme dönüştürülür. Daha sonar tekrarlama bağıntısı ve diferansiyel dönüşüm katsayıları elde edilir. Son olarak, üç örnek için bu denklemin oldukça doğru yaklaşık çözümleri bulunur.

Anahtar kelimeler: Singuler Pertürbe Burger-Huxley Kısmi Diferansiyel Denklem, Yaklaşık Çözüm, Sınır Katı, RDTM.

1. Introduction

Burgers-Huxley equation has a serious importance in class of nonlinear partial differential equations. Burgers-Huxley equation defines the interactions between reaction mechanisms, convection effects and diffusion transports [10]. Furthermore, Burgers-Huxley equation is successfully applied to describe some ecological models. Firstly, this equation was described by Bateman [11] and then utilized by Burgers [9] in a mathematical modelling in turbulence. Burger-Huxley equation has studied by using a variety of methods such as Adomian decomposition method [3,4], meshless method [6], finite difference method [5,21], comparative study of some numerical methods as nonstandard finite difference method; explicit exponential finite difference method; fully implicit exponential finite difference method [8,22], Strang Splitting Method [23] etc. Exact solution of Burgers-Huxley equation was studied [7]. We know that most of the standard numerical methods can easily approach the solution of the Burger-Huxley equation without the small singular perturbation parameter. However, it is difficult to reach solution in the presence of the perturbation parameter [17]. Therefore, we use the RDTM in this study as it can eliminate this difficulty. The difference of the reduced differential transform method from the classical transformation methods is that it transforms a partial differential equation into a semi-algebraic equation. Its advantage over classical transformation methods is to reach the correct result with fewer iterations.

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The reduced differential transform method has been used in the solution of many linear and nonlinear problems [1,2,15,18]. Equations with fractional properties can also be studied with this method [24].

This study is organized as follows: The properties of the singularly perturbed Burger-Huxley equation are given, respectively. The reduced differential transform method is introduced. Finally, the reduced differential transform method is applied to sample problems with only a few iterations. In the series solution, as a result of this application, the different values given to x and t are presented with figures of the approximate solution values.

In this paper, we examine the following singularly perturbed Burger-Huxley differential equation via RDTM [5]:

$$\begin{aligned} &-\varepsilon\frac{\partial^2 u}{\partial x^2}+\alpha u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}-\beta u(1-u)(u-\gamma)=0,\\ &u(x,0)=f_0(x),\ x\in(0,1),\\ &u(0,t)=f_1(t),\ u(1,t)=f_2(t),\ t\in(0,1], \end{aligned} \tag{1}$$

where $\alpha\geq 0,\ \beta\geq 0,\ \gamma\in(0,1);\ f_0(x),\ f_1(t)$ and $f_2(t)$ are continuous functions and $0<\varepsilon\ll 1$ is singular perturbation parameter. This problem usually has one or two boundary layers.

2. Material and Method

The two-dimensional differential transform method was first applied to two-dimensional partial differential equations according to t and x variables in 1999 by Chen and Ho [12]. On the other hand, the reduced differential transform method, which is applied only according to the $t-$ time or $x-$ space variables. It is also preferred in the solution of linear and nonlinear partial differential equations by performing a small number of iterations and reaching a solution in a short time. Using the definitions and properties of the differential transform method, we can write the form of the reduced differential transform method [12-14,16,19]:

The differential transformation function $U(h,k)$ corresponding to the two-component function $u(x,t)$ considering $t-$time variable [20], it is defined as

$$U_h(x)=\frac{1}{h!}\left[\frac{\partial^h u(x,t)}{\partial x^h}\right]_{t=0}, \tag{2}$$

$$u(x,t)=\sum_{h=0}^{\infty}U_h(x)\,t^h=U_0(x)+U_1(x)t+U_2(x)t^2+U_3(x)t^3+\cdots. \tag{3}$$

Some features of the reduced differential transformation method (2)-(3) respect to the $t-$time variable are given in the following theorems [1,2,15].

Teorem 1. $v(x,t)=\dfrac{\partial^2 u}{\partial x^2}\rightarrow V_h(x)=\dfrac{d^2U_h(x)}{dx^2}.$

Theorem 2. $v(x,t)=\dfrac{\partial u}{\partial x}\rightarrow V_h(x)=\dfrac{dU_h(x)}{dx}.$

Theorem 3. $v(x,t)=\dfrac{\partial u}{\partial t}\rightarrow V_h(x)=(h+1)U_{h+1}(x).$

Theorem 4. $v(x,t)=u\dfrac{\partial u}{\partial x}\rightarrow V_h(x)=\displaystyle\sum_{s=0}^{h}\frac{dU_s(x)}{dx}U_{h-s}(x).$

Theorem 5. $v(x,t)=u^2\rightarrow V_h(x)=\displaystyle\sum_{s=0}^{h}U_s(x)U_{h-s}(x).$

Theorem 6. $v(x,t)=u^3 \to V_h(x)=\sum_{m=0}^{h}\sum_{r=0}^{m}U_r(x)U_{m-r}(x)U_{h-m}(x).$

These theorems will be used in solving the singularly perturbed Burger-Huxley equation.

2.1. Applications of RDTM

Now, we present advantages and effectiveness of our method on three examples.

Example 1. As a special case of the equation (1), we consider the following singularly perturbed Burger-Huxley equation:

For $\alpha=1,\ \beta=1,\ \gamma=\frac{1}{2},\ f_0(x)=x(1-x^2),\ f_1(t)=0,\ f_2(t)=0,$

$$
\begin{aligned}
&-\varepsilon\frac{\partial^2 u}{\partial x^2}+u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}-u(1-u)\left(u-\frac{1}{2}\right)=0,\\
&u(x,0)=x(1-x^2),\ x\in(0,1),\\
&u(0,t)=u(1,t)=0,\ t\in(0,1].
\end{aligned}
\tag{4}
$$

There is no exact solution to this problem. Therefore, we will find an approximate solution.
Using the reduced differential transform method, the differential transforms corresponding to each term and initial condition according to the $t-$time variable in the singularly perturbed Burger-Huxley equation (5) are as follows:

$$
\left.
\begin{aligned}
&\varepsilon\frac{\partial^2 u}{\partial x^2}\to\frac{d^2U_h(x)}{dx^2},\\
&u\frac{\partial u}{\partial x}\to\sum_{s=0}^{h}U_{h-s}(x)\frac{dU_s(x)}{dx},\\
&\frac{\partial u}{\partial t}\to(h+1)U_{h+1}(x),\\
&u^2\to\sum_{s=0}^{h}U_s(x)U_{h-s}(x),\\
&u^3\to\sum_{m=0}^{h}\sum_{r=0}^{m}U_r(x)U_{m-r}(x)U_{h-m}(x),\\
&u(x,0)=x(1-x^2)\to U_0(x)=x(1-x^2).
\end{aligned}
\right\}
\tag{5}
$$

If the differential transforms (5) are written in equation (4), the following recurrence relation is obtained:

$$
U_{h+1}(x)=\frac{1}{h+1}\left(
\begin{aligned}
&\varepsilon\frac{d^2U_h(x)}{dx^2}-\sum_{s=0}^{h}U_{h-s}(x)\frac{dU_s(x)}{dx}-0.5U_h(x)+\frac{3}{2}\sum_{s=0}^{h}U_s(x)U_{h-s}(x)\\
&-\sum_{m=0}^{h}\sum_{r=0}^{m}U_r(x)U_{m-r}(x)U_{h-m}(x)
\end{aligned}
\right).
$$

Here, the following differential transformation coefficients with five iterations for values $h=0,1,2,3$ are given:

$$U_0(x) = x(1-x^2),$$

$$U_1(x) = -6\varepsilon x - 15x + 35x^3 + \frac{3}{2}x^2 - 3x^4 + \frac{3}{2}x^6 - 3x^7 + x^9,$$

$$U_2(x) = 1875000000x - 4875000000x^2 + \frac{3}{2}\varepsilon + 1800000000\varepsilon x + 1725000000x^4 - 6.375000000x^3$$

$$-6000000000x^5 - 1237500000x^6 + 3225000000x^7 - 2325000000x^9 - \frac{45}{4}x^8 - 27\varepsilon x^2$$

$$+\frac{63}{2}\varepsilon x^4 - 81\varepsilon x^5 + 45\varepsilon x^7 - 3\varepsilon x^3 - \frac{9}{2}x^{11} + 15x^{10} - \frac{15}{4}x^{12} + \frac{15}{2}x^{13} - \frac{3}{2}x^{15}.$$

These differential transform coefficients are written in equation (3) and the approximate solution $u_{RDTM}(x,t)$ is obtained as follows

$$u_{RDTM}(x,t) = \sum_{h=0}^{\infty} U_h(x)t^h = U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + U_4(x)t^4 + U_5(x)t^5.$$

$$= x - x^3 - 6t\varepsilon x - 15tx + 35tx^3 + \frac{3}{2}tx^2 - 3tx^4 + \frac{3}{2}tx^6 - 3tx^7 + tx^9.$$

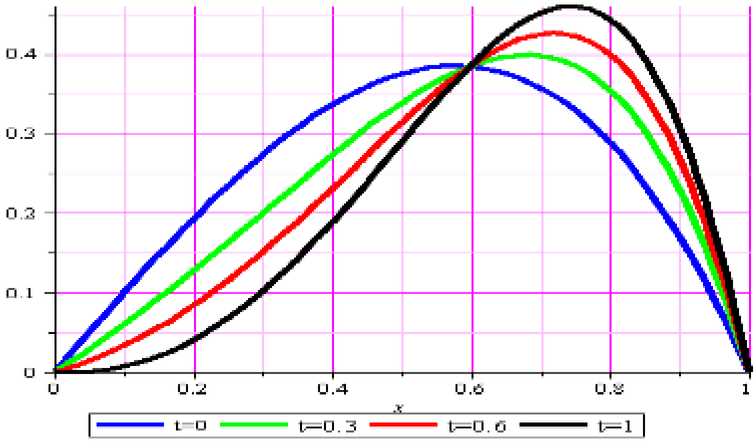

Figure 1. The approximate solution curves obtained with RDTM for $\varepsilon = 2^{-15}$

Example 2.
As a special case of the equation (1), we solve the following singularly perturbed Burger-Huxley equation:

$$-\varepsilon\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0,$$
$$u(x,0) = x(1-x^2),\ x \in (0,1), \tag{6}$$
$$u(0,t) = u(1,t) = 0,\ t \in (0,1],$$

where $\alpha = 1,\ \beta = 0,\ f_0(x) = x(1-x^2),\ f_1(t) = 0,\ f_2(t) = 0.$

The exact solution is not available. Thus, we will find an approximate solution for the singularly perturbed Burger-Huxley equation (6) via the reduced differential transform method.
Now, we find recurrence relation as

$$U_{h+1}(x) = \frac{1}{h+1}\left(\varepsilon\frac{d^2U_h(x)}{dx^2} - \sum_{s=0}^{h} U_{h-s}(x)\frac{dU_s(x)}{dx}\right).$$

Then, we have differential transform coefficients as

$$U_0(x)=x(1-x^2),$$
$$U_1(x)=-6\varepsilon x-x+4x^3-3x^5,$$
$$U_2(x)=18\varepsilon x-42\varepsilon x^3+x-10x^3+21x^5-12x^7,$$
$$U_3(x)=120x^7-55x^9+20x^3-84x^5-36\varepsilon x+252\varepsilon x^3-96\varepsilon^2 x-288\varepsilon x^5-x.$$

Finally, we reach the following approximate solution $u_{RDTM}(x,t)$ with four iterations

$$u_{RDTM}(x,t)=x-x^3-6t\varepsilon x-tx+4tx^3-3tx^5+18t^2\varepsilon x-42t^2\varepsilon x^3+t^2x-10t^2x^3+21t^2x^5-12t^2x^7.$$

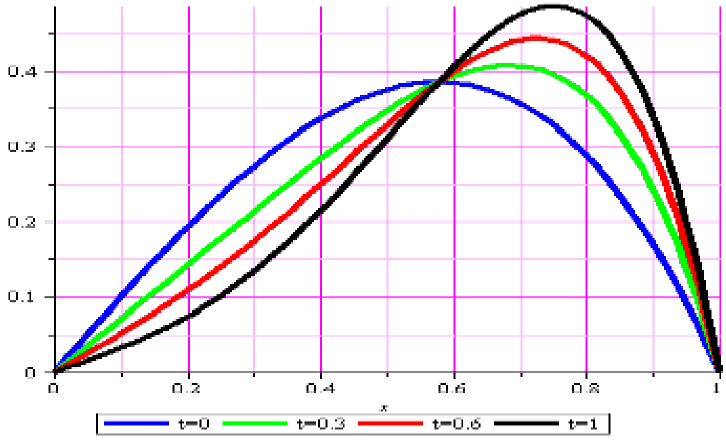

Figure 2. Comparison of the approximate solution curves obtained with RDTM for $\varepsilon=2^{-15}$

Example 3.

$$-\varepsilon\frac{\partial^2u}{\partial x^2}+u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}-u(1-u)\left(u-\frac{1}{2}\right)=0,$$
$$u(x,0)=\sin(\pi x),\ x\in(0,1), \tag{7}$$
$$u(0,t)=u(1,t)=0,\ t\in(0,1],$$

where $\alpha=1,\ \beta=1,\gamma=\frac{1}{2},\ f_0(x)=\sin(\pi x),\ f_1(t)=0,\ f_2(t)=0.$

The exact solution of (7) is not available. So, we benefit the following reduced differential transform method procedure:

Firstly, we can give recurrence relation

$$U_{h+1}(x)=\frac{1}{h+1}\left(\begin{aligned}&\varepsilon\frac{d^2U_h(x)}{dx^2}-\sum_{s=0}^{h}U_{h-s}(x)\frac{dU_s(x)}{dx}-0.5U_h(x)+\frac{3}{2}\sum_{s=0}^{h}U_s(x)U_{h-s}(x)\\&-\sum_{m=0}^{h}\sum_{r=0}^{m}U_r(x)U_{m-r}(x)U_{h-m}(x)\end{aligned}\right).$$

From the above recurrence relations for $h=0,1,2,3,$ we obtain differential transformation coefficients with four iterations. Then, using the equation (3), we have the approximate solution as:

$$U_0(x)=\sin(\pi x),$$
$$U_1(x)=-\varepsilon\pi^2\sin(\pi x)-\pi\sin(\pi x)\cos(\pi x)-0.5\sin(\pi x)+\frac{3}{2}\sin^2(\pi x)-\sin^3(\pi x),$$

...

Finally, using above relations and formula (3), we reach approximate solution of the problem (6) with three iterations as follows:

$$u_{RDTM}(x,t) = 2000000000.10^{(-9)} sin(3141592654x)(50000000010^{8} - 493480220210^{9} t\varepsilon$$
$$-1570796327.10^{9} tcos(3141592654x) - 75000000010^{8} t$$
$$+75000000010^{8} tsin(3141592654x) + 50000000010^{8} tcos(3141592654x)^{2}).$$

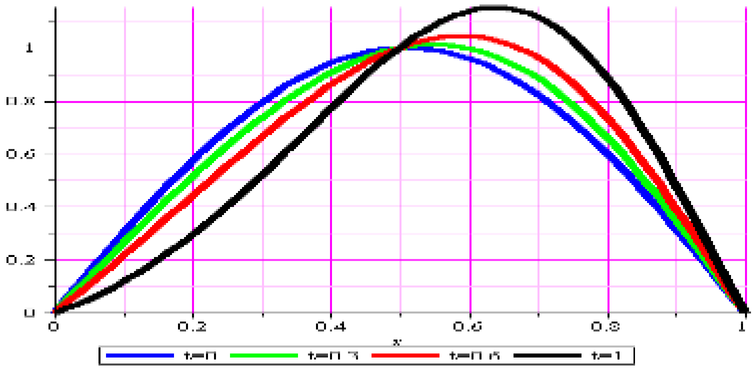

Figure 3. Comparison of the approximate solution curves obtained with RDTM for $\varepsilon = 2^{-15}$

In all three examples, there is a boundary layer at $x = 1$.
As seen in figures, as values of t increase, approximate solution curves get closer to the coordinate axis around the boundary layer of $x = 1$. We plotted these results in Figure 1, 2, 3 for comparing RDTM solutions with respect to different the $t -$time variables. Thus, it is clear that the approaches obtained using the proposed method is in high accuracy.

3. Remark

The exact solution of the singularly perturbed Burger-Huxley partial differential equation in this study cannot be seen in the literature. Thus, this paper will give an important idea to determine approximate solutions behavior of these problems via RDTM.

4. Conclusion

We considered an efficiently method for solving singularly perturbed Burger-Huxley partial differential equation. The solutions were very rapidly convergent by using this method. The numerical results have been obtained by mathematics computer programe and shown in figures. The numerical values in all figures shown that we arrived at a correct numerical approach.

Author’s Contributions

All contributions belong to the author in this paper.

Statement of Conflicts of Interest

No potential conflict of interest was reported by the author.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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