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Matrix Transformations and Generalized Almost Convergence

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Matrix Transformations and Generalized Almost Convergence

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ABSTRACT

Recently M. Stieglitz introduced the concept of F_β –convergence of sequence and generalized the results of Eizen and Laush, King, and Shaefer for more general classes of matrices. Quite recently Ahmad and Mursaleen have extended the space F_β of F_β – convergent sequences to F_β (p). In the present paper we furnished a set of necessary and sufficient conditions for each $(c_o(p), F_{o\beta}(p))$, $(1(p), F_\beta)$ and $(M_o(p), F_\beta)$ matrices.

1. INTRODUCTION

In 1948, Lorentz [2] introduced the concept of almost convergence. In 1973, recently M. Stieglitz [6] generalized this concept of almost convergence to F_β – convergence. And quite recently Ahmad and Mursaleen [1] extended this space F_β of F_β -convergent sequences to F_β (p) just as c, c_o, and f were extended to c (p), c_o (p) and f (p) respectively. For real $p_n > o$ and $\sup p_n < \infty$, we have [1],

$$F_\beta(p) = \{ x : \lim_n |(B_i x)_n - Le|^{p_n} = 0, \text{ uniformly in } i, \text{ for some } L\}$$

$$F_{o\beta}(p) = \{ x : \lim_n |(B_i x)_n|^{p_n} = 0, \text{ uniformly in } i \}$$

In particular, if $p_n = p > 0$ for every n, we have

$F_\beta(p) = F_\beta$ and $F_{o\beta}(p) = F_{o\beta}$. If we put $\beta = \beta_o$

F_β (p) reduces to c (p) and if $\beta = \beta_1$, $F_\beta(p) = f(p)$.

2. In this note, we prove the following theorems:

Theorem 2.1. $A \varepsilon (c_o(p), F_{o\beta}(p))$ if and only if

(i) $N(A) < \infty$ and there exist $r \geq 0$ and $B > 1$ such that

$$\sup_{\substack{0 \leq i < \infty \\ r \leq n < \infty}} \left\{ \sum_l \left| \sum_k b_{nk}(i)\, a_{kl} \right| B^{-1/p_l} \right\}^{p_n} < \infty$$

(ii) $\lim_n \left| \sum_l b_{nl}(i)\, a_{lk} \right|^{p_n} = 0$ (uniformly in i, k fixed).

Proof. Necessity. Suppose $A \varepsilon (c_o(p), F_{o\beta}(p))$. We know the fact $A: c_o \to m$, hence $N(A) < \infty$. Define $e_k = \{\delta_{jk}\}$, where

$$\delta_{jk} = \begin{cases} 0 & (j \neq k), \\ 1 & (j = k). \end{cases}$$

Since $e_k \varepsilon c_o(p)$, now

$$T^{(k)}_{in} = \sum_l b_{nl}(i) \sum_k a_{lj}\, \delta_{jk}$$

$$= \sum_l b_{nl}(i)\, a_{lk}$$

Therefore, for all $k \geq 1$, $\delta_{jk} \to o$ as $j \to \infty$, it follows that

$$\lim_n \left| T^{(k)}_{in} \right|^{p_n} = 0$$

This is equivalent to (ii). Now, put

$$f_m(x) = \left| \sum_{l=0}^{m} \left(\sum_k b_{nk}(i)\, a_{kl} \right) x_l \right|^{P_n}$$

It is easy to see that $\{f_m(x)\}$ is a sequence of continuous linear functionals such that $\lim_m f_m(x)$ exists. We note that

$$Tx = \left| (B_i(Ax))_n \right|^{P_n} = \lim_m f_m(x).$$

Therefore, by virtue of Banach-Steinhaus theorem, it follows that $T \varepsilon c'_o(p)$ (continuous dual space of $c_o(p)$) and $\| T \| < \infty$. Let us define for each r:

$$y^{(r)}_l = \begin{cases} \delta^{k/p_l} \operatorname{sgn}\left(\sum_k b_{nk}(i)\, a_{kl} \right) & (0 \leq l \leq r), \\ 0 & (\text{otherwise}). \end{cases}$$

where K is a constant. Then it follows that

$$y_l^{(r)} \in c_o(p)$$

and

$$\left\{ \sum_{l=1}^{r} \left| \sum_k b_{nk}(i)\, a_{kl} \right| B^{-1/p_l} \right\}^{p_n} \leq K$$

for eâch n and r, where $B = \delta^{-K}$. Therefore (i) holds.

Sufficiency. Let us suppose that the conditions (i) and (ii) hold and that $x \in c_o(p)$. For $C = \max(1, 2^{H-1})$ where $H = \sup p_n$, we have the inequality (see Maddox [5], p. 346)

$$| B_i(Ax)_n |^{p_n} \leq C(I_1 + I_2)$$

where

$$I_1 = \left| \sum_{l \leq l_o} \left(\sum_k b_{nk}(i)\, a_{kl} \right) x_l \right|^{p_n},$$

and

$$I_2 = \left| \sum_{l > l_o} \left(\sum_k b_{nk}(i)\, a_{kl} \right) x_l \right|^{p_n}$$

l and n both are larger than l_o.

Since (ii) holds, therefore, there exists $n_o > o$ such that $n > n_o$,

$$\left| \sum b_{nl}(i)\, a_{lk} \right|^{p_n \, 1/p_n} < \varepsilon, \text{ uniformly in i.}$$

Therefore, for ruch n

$$\text{(I)} \quad I_1 < \left(\sum_{l \leq l_o} \left| \left(\sum_k b_{nk}(i)\, a_{kl} \right) x_l \right|^{p_n} \right)$$

$$< \varepsilon \left(\sum_{l \leq l_o} | x_l |^{p_n} \right) \text{ uniformly in i.}$$

Again for $n > n_o$,

$$\text{(II)} \quad I_2 \leq \left(\sum_{l > l_o} \left| \left(\sum_k b_{nk}(i)\, a_{kl} \right) x_l \right| \right)^{p_n} < \varepsilon \text{ uniformly in i.}$$

Hence the sufficiency follows from (I) and (II).

This completes the proof.

Theorem 2.2. $A \in (l(p), F_\beta)$ if and only if

(i) There exists $B > 1$ such that for every i

$$\sup_n \sum_k |C(n,k,i)|^{q_k} B^{-q_k} < \infty \quad (1 < p_k < \infty)$$

$$\sup_{n,k} |C(n,k,i)|^{p_k} < \infty, \quad (0 < p_k \leq 1) \quad (p_k^{-1} + q_k^{-1} = 1)$$

where

$C(n,k,i) = \Sigma_l \, b_{nl}(i) \, a_{lk}$

(ii) $\lim_n C(n,k,i) = \alpha_k$ (uniformly i, k fixed).

Proof. Necessity. We only consider the case $1 < p_k < \infty$. The case $0 < p_k \leq 1$ has a similar proof. Let $A \in (l(p), F_\beta)$. Define $e_k = (0,0, \ldots, 0, 1, 0, \ldots)$. Since $e_k \in l(p)$, (ii) must hold. Now $(B_i(Ax))_n$ exists for each n and $x \in l(p)$. If we put $T_{n,i}(x) = (B_i(Ax))_n$, then $\{T_{n,i}(x)\}_n$ is a sequence of continuous real functions on $l(p)$ and further $\sup_n |(B_i(Ax))_n| < \infty$ on $l(p)$. Now by uniform boundedness principle (see Lascarides and Maddox [3]) the necessity follows.

Sufficiency. For every $j \geq 1$, we have

$$\sum_{k=1}^{j} |C(n,k,i)|^{q^k} B^{-q_k} \leq \sup_n \sum_k \left|\sum_l b_{nl}(i) \, a_{lk}\right|^{q_k} B^{-q^k}$$

therefore

$$\sum_k |\alpha_k|^{q_k} B^{-q_k} = \lim_{j \to \infty} \lim_{n \to \infty} \sum_{k=1}^{j} |C(n,k,i)|^{q_k} B^{-q_k}$$

$$\leq \sup_n \sum_k |C(n,k,i)|^{q_k} B^{-q_k} < \infty.$$

Thus the series $\Sigma_k C(n,k,i) x_k$ and $\Sigma_k \alpha_k x_k$ converge (see Maddox [4]) for each n and $x \in l(p)$. Now, for $\varepsilon > 0$ and $x \in l(p)$, choose k_0 such that

$$\text{(I)} \qquad \left(\sum_{k=k_o+1}^{k_o} | x_k |^{p_k}\right)^{1/H} < \infty .$$

By (ii) there exists n_o such that

$$\left| \sum_{k=1}^{k_o} (C(n,k,i) - \alpha_k) \right| < \varepsilon \ \forall \ n > n_o .$$

Since (i) hods, it follows that (see Lascarides and Maddox [3]).

$$\left| \sum_{k=k_o+1}^{\infty} C(n, k, i) - \alpha_k \right|$$

is arbitrarily small. Therefore,

$\lim_n \Sigma_k C(n, k,i) x_k = \Sigma_k \alpha_k x_k$

uniformly in i. This completes the proof.

Corollary. A ε (l (p), $F_{o\beta}$) if and only if

(i) Condition (i) theorem (2.2) holds,

(ii) $\lim_{n \to \infty} C(n, k,i) = 0$ uniformly in i.

We now characterise the matrices in the class (M_o (p), F_β). For $p_k > 0$ we define (see Maddox [4]),

$$M_o(p) = \bigcup_{B>1} \{ x : \sum_k | x_k | B^{1/pk} < \infty \} .$$

When $p_k = p \ \forall \ k$, we have $M_o(p) = l_1$. Also $M_o(p) = l_1$ for $\inf p_k > 0$.

Theorem 2.3. A ε (M_o (p), F_β) if and only if

(i) For every integer $B > 1$,

$$\sup_{n,k} | C(n, k, i) | B^{1/p_k} < \infty \ (\forall \ i)$$

(ii) $\lim_n C(n, k, i) = \alpha_k$ (uniformly in i, k fixed)

Proof. Necessity. Suppose A ε (M_o (p), F_β). Since e_k ε M_o (p), (ii) holds. On contrary let us suppose that (i) is not true, then ∃ $B > 1$ such that

$$\sup_{n,k} | C(n, k, i) | B^{1/p_k} = \infty$$

So by theorem (2.2), $C = (C_{nk}) = (a_{nk} B^{1/p_k}) \notin (l_1, F_\beta)$ that is, $\exists\, x \varepsilon L_1$ such that $Cx \notin F_\beta$. Now $y = (y_k) = (B^{1/pk} x_k) \varepsilon M_o$ (p), but $Ay = Cx \notin F_\beta$, which contradicts that $A \varepsilon (M_o(p), F_\beta)$.

Sufficiency. Suppose that the conditions (i) and (ii) hold and $x \varepsilon M_o$ (p). Then

$$|\Sigma C(n, k, i) x_k | \leq \Sigma_k | x_k | B^{-1/pk} | C(n, k, i) | B^{1/pk} < \infty .$$

Now similarly as in Theorem (2.2) we have

$$\lim_{n \to \infty} \Sigma_k C(n, k, i) x_k = \Sigma_k a_k x_k$$

uniformly in i and hence $A \varepsilon (M_o(p), F_\beta)$.

This completes the proof.

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