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by

Fikri AKDENİZ

7

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Fikri AKDENİZ*

ABSTRACT

In this paper we are concerned with a comparison of two estimators of the macro parameters. To do that we axamine the covariance structure of each estimator to determine which estimator has the smaller variance.

1. INTRODUCTION

Theil (1954) discussed certain relationship between the micro and macro parameters. The problem also discussed by Kloek (1961) Mısra (1969) Lütjohann (1972), Wu (1973), Akkina (1974), and Sasaki (1978).

We are concerned with a comparison of two estimators of the macro parameters. One obvious estimate is obtained directly from the application of ordinary least squares to the macro equation. It is well known that the expectation of this estimate generally differs from the aggregate of the expectations of ordinary least squares estimates ol tained from the micro equations. Therefore, comparing these two estimators will necessarily involve considerations of bias as well as efficiency. Nevertheles this is the comparison which has traditionally been made.

In the present study the direct ordinary least squares estimate of the macro equation is retained, but a rather different aggregate of the micro estimates is employed. Instead of simply aggregating the micro estimates, the projection into the space spanned by the macro explanatory variables is taken.

* University of Ankara, Faculty of Secience, Dept. of Mathematics, Ankara, TURKEY.

In section 2, we specify the micro and macro models. Section 3 is devoted to comparing two estimators of the macro parameters.

2. NOTATION AND ASSUMPTIONS

Let the economic relationship for the i-th economic unit be given by

(2.1) $Y_i = X_i \beta_i + U_i$, i = 1,2,...,N

where Y_i and U_i are each a $T \times 1$ vector, X_i is a $T \times K$ matrix and β_i is a $K \times 1$ vector of the unknown constants.

The following assumptions are used throughout this paper:

ASSUMPTION 1. The matrix X_i has rank $r_i \leq K$.

ASSUMPTION 2. $E(U_i / X_i) = 0$, $E(U_i U'_j / X_i, X_j) =$

$$\begin{cases} \Sigma_{ii} & i=j \\ 0 & i \neq j \end{cases}$$

By arranging the vectors and matrices of all micro units in the same way as in Lütjohann (1972), the micro model can be written as

(2.2) $Y^* = X^* \beta^* + U^*$, $E(U^* / X^*) = 0$, $E(U^* U^{*'} / X^*) = \Sigma$.

where Y^* and U^* are each a $NT \times 1$ vector, X^* a $NT \times NK$ block-diagonal matrix with $T \times K$ matrices on the diagonal and zeros elsewhere and β^* a $NK \times 1$ vector of the unknown micro parameters. The matrix Σ is block-diagonal matrix, consisting of the matrices Σ_{ii} which have generally unknown elements.

If corresponding micro observations are linearly aggregated over N units the macro observations can be expressed by $Y = J_T Y^*$ a $T \times 1$ vector and $X = J_T X^* J'_K$ a $T \times K$ matrix, where

$$J_T = I_T \otimes j'_N \quad , \quad J_K = I_K \otimes j'_N$$

I_T is a $T \times T$ identity matrix, j_N a $N \times 1$ column vector of unit elements and $A \otimes B$ denotes the Kronecker product of the matrices A and B (Graybill (1969)). Using these aggregate observations the macro model can be written as

(2.3) $Y = X \beta + U$.

For less than full rank model, $X\beta$ is defined as

$$X\beta = E(XX^-Y \mid X^*) = E(X\hat{\beta} \mid X^*)$$

$$= X \sum_{i=1}^{N} X^- X_i \beta_i = X \sum_{i=1}^{N} \Gamma_i \beta_i.$$

where X^- denotes the Moore-Penrose generalized inverse of X

(Graybill (1969)), $\Gamma_i = (X'X)^- X'X_i = X^- X_i$ and $\sum_{i=1}^{N} \Gamma_i = X^- X$

(see Kloek (1961) and Sasaki (1978)). U is a $T \times 1$ vector of unobservable macro errors, and we assume that the rank of X is equal to r ($r \leq K$). On the other hand, it is shown that the macro-relations disturbance vector is

$$(2.4) \quad U = (I-XX^-) J_T X^* \beta^* + J_T U^* = \sum_{i=1}^{N} (I-XX^-) X_i \beta_i + \sum_{i=1}^{N} U_i$$

$$= S \beta^* + J_T U^* = \sum_{i=1}^{N} V_i \beta_i + \sum_{i=1}^{N} U_i$$

where $S = J_T X^* - X^- P$ and $P = X^- J_T X^*$ is a $K \times NK$ matrix of least squares coeffecients, and $V_i = (I-XX^-) X_i$, $\sum_{i=1}^{N} V_i = 0$.

In equation (2.4) there are two terms: the first term is the linear combition of all micro coeffecients, and the second is the sum of corresponding micro errors (see Wu (1973)).

3. A COMPARISON OF TWO ESTIMATORS OF THE MACRO PARAMETERS

Lemma 3.1 The system of equations relating to β and β^*

$$X \beta = XX^- J_T X^* \beta^*$$

is consistent. Proof is given by Akdeniz and Milliken (1975).

By using the relationship between β and β^* one can obtain two estimators of $X\beta$. The least squares estimate of $X\beta$ from (2.3) is $X\beta = XX^-Y$. But, another estimator exists by using the relationship $X\beta = XX^- J_T X^* \beta^*$. Let b^* be the vector of least squares

estimate of the micro parameters obtained from (2.2), then an estimate of $X\beta$ is

(3.1) $Xb = XX^-J_TX^*b^*.$

The expectation of Xb is

$$(3.2)\ R\,(Xb\ /\ X^*) = E\,(XX^-J_1X^*b^*\ /\ X^*)$$

$$= XX^-J_TX^*X^{*-}X^*\beta^*$$

$$= XX^-J_TX^*\beta^*$$

$$= X\beta.$$

Thus we have two unbiased estimators of $X\beta$, $X\hat{\beta}$ and Xb. If X is not of full column rank, $X\hat{\beta}$ and Xb are uniquely determined. These two estimators will, in general, not be identical, but the following theorem presents necessary and sufficent conditions under which they are equivalent.

Theorem 3.1 The two estimators of $X\beta$, $X\hat{\beta}$ and Xb are equal if and only if $C\,(X_1) = \ldots = C\,(X_N) = C\,(X)$, where $C\,(X_i)$ denotes the column vector space spanned by the colums of X_i.

Proof: The two estimators can be expressed in terms of the micro dependent variables as

(3.3) $X\,\beta = XX^-J_TY^* = XX^-Y$ and $Xb = XX^-J_TX^*b^* = $

$XX^-J_TX^*X^{*-}Y^*.$

The difference of the two estimators is

$$XX^-J_TY^* - XX^-J_TX^*X^{*-}Y^* = XX^-J_T\,(I - X^*X^{*-})\,Y^*.$$

Assuming that $Xb = X\hat{\beta}$ implies that

$$XX^-J_T\,(I - X^*X^{*-})\,Y^* = 0$$

for all possible values of Y, hence we must have

$$XX^-J_T\,(I - X^*X^{*-}) = 0.$$

That equation can be expanded into

$$X'^-X'J_T\cdot\begin{bmatrix} I-X_1X_1^- & 0 & \ldots & 0 \\ 0 & I-X_2X_2^- & \ldots & 0 \\ \cdot & & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & & \ldots & I-X_NX_N^- \end{bmatrix} = X'^-X'(I- X_1X_1^-,\ldots, I-X_NX_N^-) = 0$$

Since the matrix product is zero, each partitions is equal to zero. i.e.,

$$X'^-X' - X'^-X'X_j X_j^- = 0, \; j = 1,2,...,N.$$

By premultiplying by X', we get $X' = X'X_jX_j^-$ and therefore $C(X_j) = C(X)$ for each $_j$. Obviously then $C(X_i) = C(X_j)$ for each i and j.

Next, assume that all of the column spaces are identical. That implies that for each pair X_i, X_j there exist $K \times K$ matrices H_{ij} and H_{ji} such that $X_i = X_j H_{ij}$ and $X_j = X_i H_{ji}$. The matrix X can also be expressed as linear combinations of the columns of each X_i as

$$\begin{aligned}
X &= X_1 + X_2 + ... + X_i + ... + X_N, \\
&= X_i H_{1i} + X_i H_{1i} + ... + X_i + ... + X_i H_{Ni} \\
&= X_i (H_{1i} + H_{1i} + ... + I + ... + H_{Ni}) \\
&= X_i A_i.
\end{aligned}$$

The matrix product $X'^-X'J_T(I - X^*X^{*-})$ can then be expressed as

$$\begin{aligned}
X'^- & \{X'(I-X_1X_1^-),...,X'(I-X_iX_i^-),...,X'(I-X_NX_N^-)\} \\
= X'^- & \{A_1'X_1'(I-X_1X_1^-),...,A_i'X_i'(I-X_iX_i^-),...,A_N'X_N' \\
& (I-X_NX_N^-)\} = 0.
\end{aligned}$$

Thus $X'^-X'J_T(I-X^*X^{*-}) = 0 = XX^-J_T(I-X^*X^{*-})Y^*$, which implies that $X\hat{\beta} = Xb$.

On the other hand, the condition of Akkina's (1974) Theorem 1 is satisfied if the condition of our Theorem 3.1 is satisfied. The experimental situations where these two estimators are identical will be the exception rather than the rule, thus we wish to determine which of the two estimators is better when they are not equal. To do that we examine the covariance structure of each estimator to determine which estimator has the smaller variance. The following theorem describes this relationship.

Theorem 3.2 The variances of the components of $X\hat{\beta}$ are smaller than or equal to the corresponding variances of the components of Xb, thus $X\hat{\beta}$ is the better of the two estimators.

Proof: The proof consists of deriving the covariance matrices of $X\hat{\beta}$ and Xb and then examining the diagonal elements of the difference of the two covariance matrices.

The covariance matrix of $X\hat{\beta}$ is denoted by Σ_1 and is defined as

$$(3.4) \quad \Sigma_1 = E(X\hat{\beta} - X\beta)(X\hat{\beta} - X\beta)'$$

$$= E(XX^-X\beta + XX^-U - X\beta)(XX^-X\beta + XX^-U - X\beta)'$$

$$= E(XX^-X\beta + XX^-U - X\beta)(XX^-X\beta + XX^-U - X\beta)'$$

$$= XX^- E(UU') XX^-.$$

We use the expression for U given in (2.4) to evaluate $E(UU')$ as

$$(3.5) \quad E(UU') = J_T \Sigma J'_T + (I-XX^-) WW' (I-XX^-),$$

where

$$W = J_T X^* \beta^* = \sum_{i=1}^{N} X_i \beta_i.$$

From the equation (2.4), we see that the expectation of U is not necessarily zero, thus the following Lemma states necessary and sufficient conditions for $E(U) = 0$.

Lemma 3.2 $E(U \mid X^*) = 0$ if and only if $C(X_1) = C(X_2) = ... = C(X_N) = C(X)$.

Proof: The proof, which consists of showing that $(I-XX^-) W =$

$$= \sum_{i=1}^{N} (I-XX^-) X_i \beta_i = \sum_{i=1}^{N} V_i \beta_i = 0 \text{ if and only if } C(X_1) = C(X_2)$$

$= ... = C(X_N) = C(X)$, is constructed similarly to that of Theorem 3.1 and is thus omitted.

It is not necessary for $E(U) = 0$ evaluate Σ_1. Thus replacing $E(UU')$ by (3.5), Σ_1 becomes

$$\Sigma_1 = XX^- \{J_T \Sigma J'_T + (I-XX^-) WW' (I-XX^-)\} XX^-$$

or

$$(3.6) \ \Sigma_1 = XX^- J_T \Sigma J'_T XX^- = XX^- \sum_{i=1}^{N} (\Sigma_{ii}) XX^-.$$

The covariance matrix of Xb is

$$(3.7) \ \Sigma_2 = E(Xb - X\beta)(Xb - X\beta)'$$
$$= E(XPb^* - XP\beta^*)(XPb^* - XP\beta^*)'$$
$$= XPX^{*-} \Sigma X'^{*-} P'X'.$$

To compare the two estimators, we examine $\Sigma_1 - \Sigma_2$ or

$$(3.8) \ \Sigma_1 - \Sigma_2 = XX^- J_T (\Sigma - X^* X^{*-} \Sigma X^* X^{*-}) (XX^- J_T)'$$
$$= XX^- J_T Q (XX^- J_T)'$$
$$= XX^- (\sum_{i=1}^{N} (\Sigma_{ii} - X_i X_i^- \Sigma_{ii} X_i X_i^-)) XX^-,$$

where $Q = \Sigma - X^* X^{*-} \Sigma X^* X^{*-}$. To substantiate the theorem, it is sufficient to show that the matrix Q is negative. Therefore, we will give the following Theorem.

Theorem 3.3 Let Σ: pxp be positive definite and let A: pxp be symmetric and idempotent matrix of rank $r \le p$, then $V = \Sigma - A \Sigma A$ is nonnegative if and only if

$$(3.9) \ A \Sigma (I-A) = 0.$$

Proof: Observe that when (3.9) holds

$$V = (I-A) \Sigma (I-A) \ge 0.$$

Since Σ is positive definite, there exists a nonsingular matrix Δ such that $\Sigma = \Delta' \Delta$ or that $\Delta'^{-1} \Sigma \Delta^{-1} = I$. Hence we obtain $V = T'T \le 0$, where $T = \Delta (I-A)$.

For the necessity part, simple computations show that if

$$S^* = A - (I-A)' \{(I-A) \Sigma (I-A)\}^- (I-A) \Sigma A,$$

or

$$S^* = A - (T'T)^- \Sigma A$$

then

$$S^{*\prime} V S^* = S^{*\prime} (\Sigma - A \Sigma A) S^* =$$
$$= - A \Sigma \{(I-A) \Sigma (I-A)\}^- \Sigma A \le 0.$$

i.e., $\Sigma - A \Sigma A$ is negative. This result completes the proof of Theorem 3.2. The sign of equality holds here only when (3.9) is true.

On the other hand, in the special case, if the variance-covariance matrices Σ_{ii}, were scalar matrices, $\sigma^2_{ii}.I$ then the $\Sigma_{ii}-A_i \Sigma_{ii}A_i = \sigma^2_{ii}(I-A_iA_i) = \sigma^2_{ii}(I-A_i)$ would be positive semi definite. (Since Σ_{ii} is positive definite and A_i is idempotent) In general, this is not true.

ÖZET

Bu çalışmada makro parametrelerin iki tahmin edicisini karşılaştıracağız. Tahmin edicilerden hangisinin daha küçük varyanslı olduğunu saptamak amacıyla her birinin kovaryans yapısını inceleyeceğiz.

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