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A LA MEMOIRE D'ATATÜRK AU CENTENAIRE DE SA NAISSANCE

On Adjointness of Comma Categories

by

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DEDICATION TO ATATÜRK'S CENTENNIAL

Holding the torch that was lift by Atatürk in the hope of advancing our Country to a modern level of civilization, we celebrate the one hundredth anniversary of his birth. We know that we can only achieve this level in the fields of science and technology that are the wealth of humanity by being productive and creative. As we thus proceed, we are conscious that, in the words of Atatürk, "the truest guide" is knowledge and science.

As members of the Faculty of Science at the University of Ankara we are making every effort to carry out scientific research, as well as to educate and train technicians, scientists, and graduates at every level. As long as we keep in our minds what Atatürk created for his Country, we can never be satisfied with what we have been able to achieve. Yet, the longing for truth, beauty, and a sense of responsibility toward our fellow human beings that he kindled within us gives us strength to strive for even more basic and meaningful service in the future.

From this year forward, we wish and aspire toward surpassing our past efforts, and with each coming year, to serve in greater measure the field of universal science and our own nation.

On Adjointness of Comma Categories

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SUMMARY

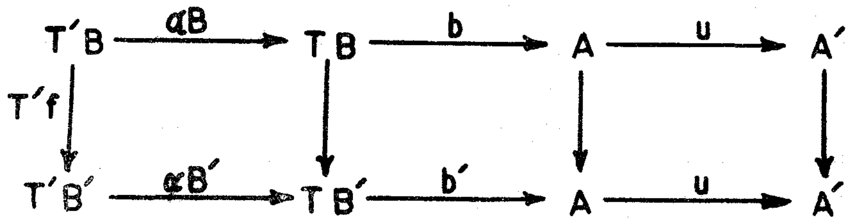
In this paper we determine a pair of adjoint functors for the categories $(T \downarrow A)$ and and $(T' \downarrow A')$ where T and T′ are functors from $\mathcal{B}$ to $\mathcal{C}$ and A and A′ are objects of $\mathcal{C}$

A precise definition of a Comma Category $(T \downarrow A)$ where T is a functor from a category $\mathcal{B}$ to another category $\mathcal{C}$ and A is an object of $\mathcal{C}$, in the sense of Maclane [2], can be stated as: The objects of $(T \downarrow A)$ are pairs (B, b) with $B \in |\mathcal{B}|$ and $b: TB \to A$ and the morphisms $h: (B,b) \to (B', b')$ are those morphisms $h: B \to B'$ in $\mathcal{B}$ for which $b' o Th = b$. In the present note we determine a pair of adjoint functors for the categories $(T \downarrow A)$ and $(T' \downarrow A')$ where, again, T′ is a functor from $\mathcal{B}$ to $\mathcal{C}$ and A′ is an object of $\mathcal{C}$.

Lemma 1: If $\alpha: T' \to T'$ is a natural transformation and $u: A \to A'$ is a morphism, then for (B, b) in $(T \downarrow A)$ the rule $(B, b) \to (B, ub\,\alpha B)$ defines a functor $R: (T \downarrow A) \to (T' \downarrow A')$

Proof: If we define R on morphisms as: for any $f: (B, b) \to (B', b')$

$R(f) = f$ such that $ub'\alpha B' o T' f = ub\, \alpha B$; that is, the following diagram commutes



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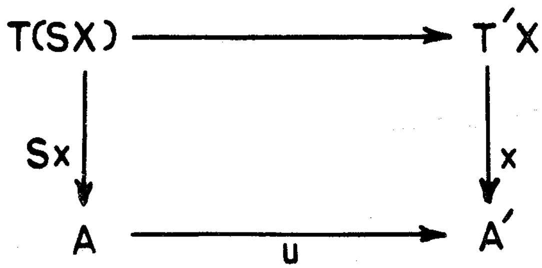
It can easily be proved that R is a functor.

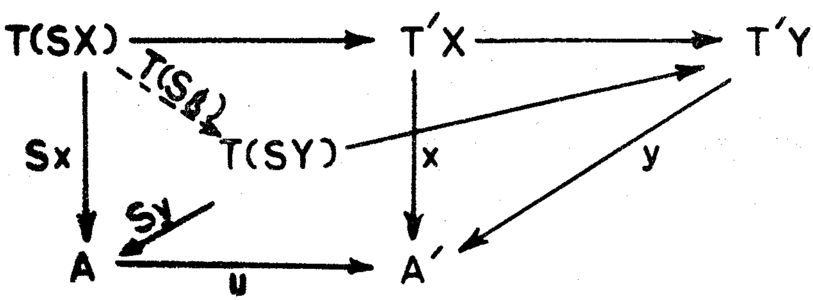
Lemma 2: If $\mathcal{C}$ is a category with pullbacks and S: (T' $\downarrow$ A') $\rightarrow$ (T $\downarrow$ A) is defined as: each (X, x) in (T' $\downarrow$ A') is associated to (SX, Sx) in (T $\downarrow$ A) such that



is a pullback, then S is a functor, if T is faithful.

Proof: Obviously S (X, x) = (SX, Sx) is an object of (T $\downarrow$ A). Let us define S on morphisms; for any g: (X, x) $\rightarrow$ (Y, y)

S (g) = Sg such that Sy T (Sg) = Sx; this can be seen in the following commutative diagram

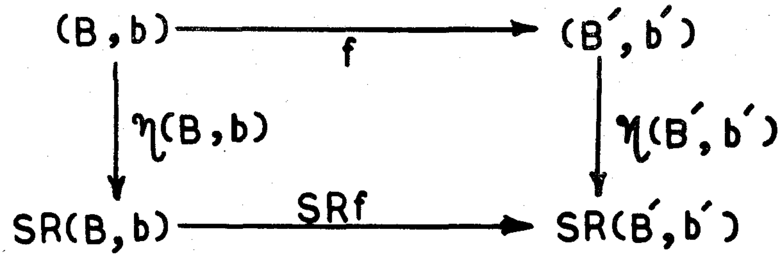


the existence of a unique TSg follows from the definition of a pullback.

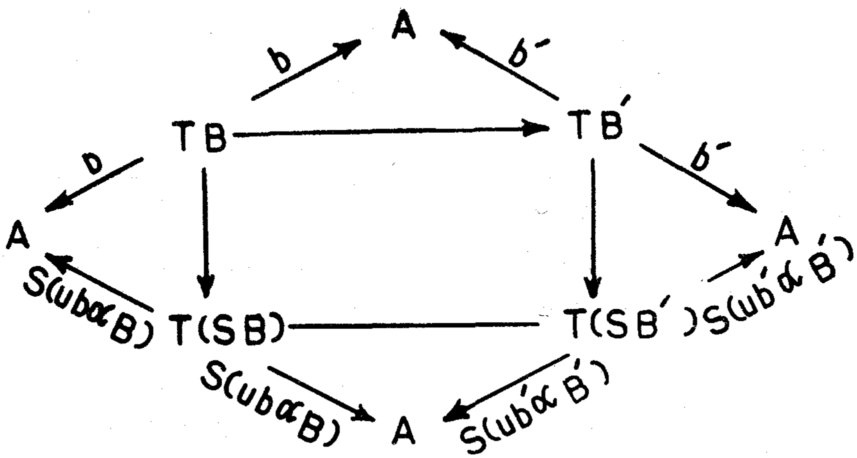
We now prove the main theorem.

Theorem: If T' is also faithful, the functors R and S defined above form a pair of adjoint functors for the categories (T $\downarrow$ A) and (T' $\downarrow$ A').

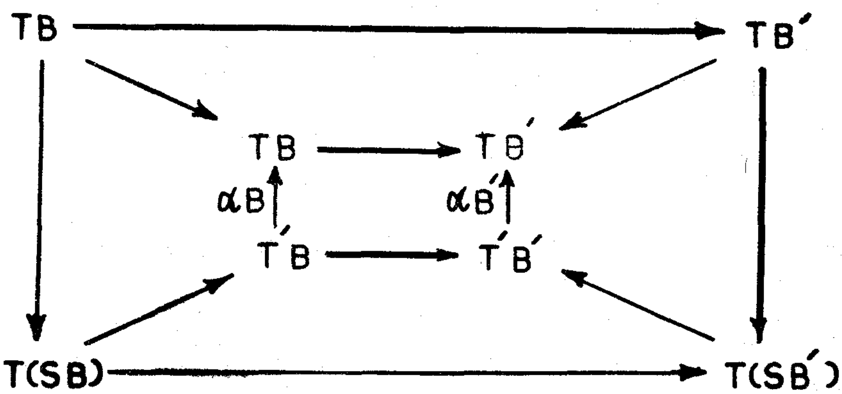
Proof: Let us define η: Id $\rightarrow$ SR and ε: RS $\rightarrow$ Id by η (B, b) = (B $\rightarrow$ SB) and ε (X, x) = (SX $\rightarrow$ X). To show that η and ε are natural trans formations, consider the diagram



This yields the following

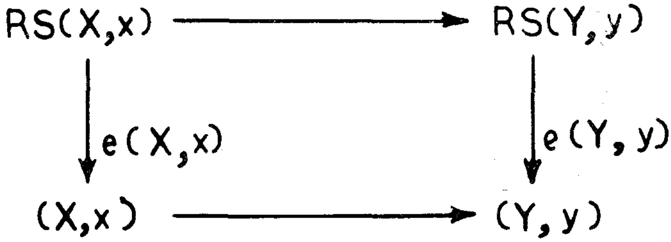


in which the outer triangles are all commutative by virtue of the definition of the comma category. Moreover, the square is also commutative as it can be broken into the following commutative diagrams.

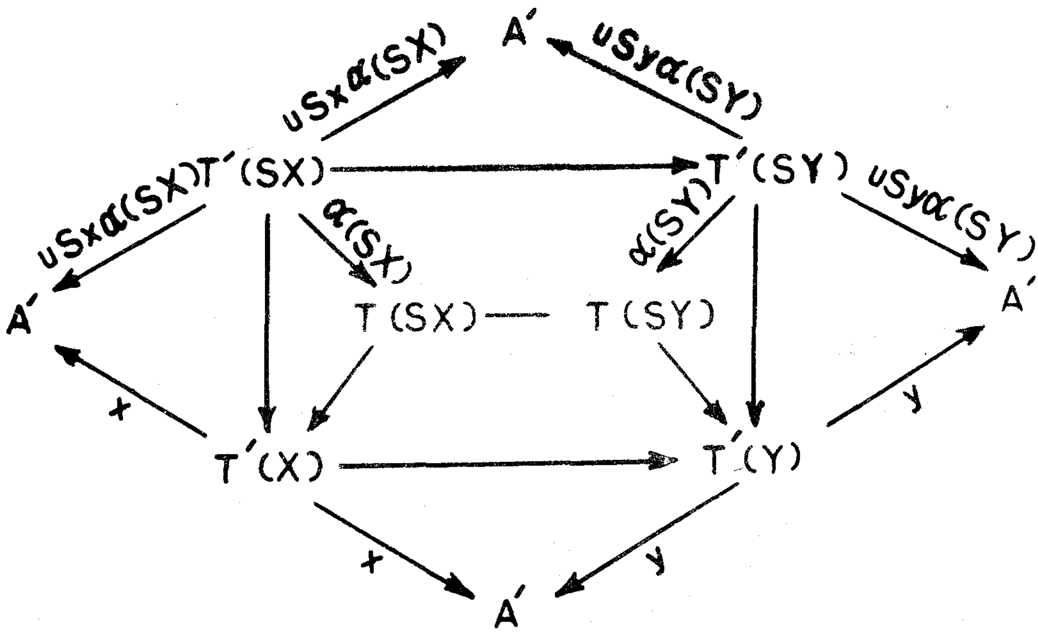


Hence η is a natural transformation.

Next, if we consider the diagram



then applying S and R we obtain the following commutatiave diagram



and hence ε is also a natural transformation.

Now considering the following composite natural transformations

$$R \xrightarrow{R\eta} RSR \xrightarrow{\varepsilon R} R \text{ and } S \xrightarrow{\eta S} SRS \xrightarrow{S\varepsilon} S$$

Wehave

$$
\begin{aligned}
(\varepsilon\ R \circ R\ \eta)\ (B, b) &= \varepsilon\ R\ (B, b) \circ R\ \eta\ (B, b) \\
&= \varepsilon\ (B, ub\alpha B) \circ R\ (B \to SB) \\
&= (SB \to B) \circ (B \to SB) \\
&= I_B \\
&= I_{(B,\ Ub\ \alpha\ B)} \\
&= I_{R\ (B,\ b)}
\end{aligned}
$$

and hence $\varepsilon\ R \circ R\ \eta = I_R$

$$
\begin{aligned}
(S\ \varepsilon \circ \eta\ S)\ (X, x) &= S\ \epsilon\ (X, x) \circ \eta\ S\ (X, x) \\
&= (S\ (SX) \to X) \circ \eta\ (SX, Sx) \\
&= (S\ (SX) \to SX) \circ (SX \to S\ (SX)) \\
&= I_{SX} \\
&= I_{(SX,\ Sx)} \\
&= I_{S\ (X,\ x)}
\end{aligned}
$$

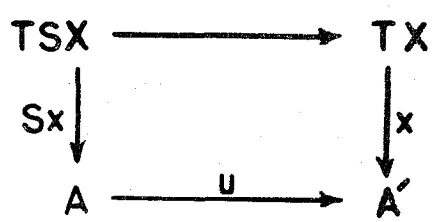
hence $S\ \varepsilon \circ \eta\ S = I_S$

This completes the proof of the theorem.

The following results determine the pair of adjoint functors for the categories $(T \downarrow A)$, $(T \downarrow A')$ and $(T \downarrow A)$, $(T' \downarrow A)$ and can be proved on similar lines.

Corollary 1: (i) If $u: A \to A'$ is a morphism in $\mathcal{C}$, then the rule $b \to ub$ defines a functor $R: (T \downarrow A) \to (T \downarrow A')$

(ii) If $\mathcal{C}$ is a category with pullbacks and each x in $(T \downarrow A')$ is associated to Sx by choosing a pullback

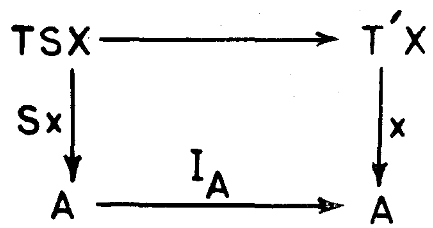


then the rule $x \to Sx$ extends to a functor $S: (T \downarrow A') \to (T \downarrow A)$ if T is faithful

(iii) The functors R and S form a pair of adjoint functors.

Corollary 2: (i) If α: T′ → T is a natural transformation then the rule b → bα B defines a functor R: (T ↓ A) (T′ ↓ A)

(ii) If $\mathcal{C}$ is a category with pullbacks and each x in (T′ ↓ A) is associated to Sx by choosing a pullback



then the rule x → Sx extends to a functor S: (T′ ↓ A) (T ↓ A) if T is faithful.

(iii) If T′ is also faithful the functors R and S form a pair of adjoint functors.

ÖZET

Bu çalışmada T ve T′ $\mathcal{B}$ kategorisinden $\mathcal{C}$ kategorisine birer funktor ve A,A′ $\mathcal{C}$ nın birer elamanı olmak üzere (T ↓ A) ve (T′ ↓ A′) kategorileri için bir adjoint funktor çifti belirlemiştir.

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