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On The Summability Field Of *l-l* Methods Of Summation

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On The Summability Field Of *l-l* Methods Of Summation

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SUMMARY

In this paper, we obtain some properties of replaceable, perfect and associative *l–l* matrix methods of summation.

I. INTRODUCTION

Let l,γ and l_∞ be the linear spaces of absolutely convergent series, convergent series and bounded sequences of complex numbers, respectively. Let $A=(a_{nk})$ be an infinite matrix and $x=(x_k)$ be a sequence of complex numbers. We write formally

(1) $$A_n(x) = \sum_k a_{nk}x_k \quad (n=1,2,...)$$

and we say that A is an *l-l* method if each series in (1) converges and $(A_n(x)) \in l$ whenever $(x_k) \in l$. Throughout the paper the sums will be taken from $k=1$ to ∞.

It is known that the necessary and sufficient condition for A to be an *l-l* method is

(2) $$\sum_n |a_{nk}| \leq M \text{ (M independent of k)}$$

[5], [6].

The matrix A is absolutely regular if and only if, it satisfies (2) and the condition

(3) $$\sum_{n=1}^{\infty} a_{nk} = 1 \ (k=1,2,...)$$

[5], [6].

Let l_A denote the summability field of A, i.e., the set of sequences which are transformed by A into l. Let l'_A be the dual of l_A. An application of Theorem 1, p. 226 and Theorem 5, p. 230 in [9] shows that l_A is an FK space, and that every $f \in l'_A$ may be evaluated as

$$(4) \qquad f(x) = \sum_n t_n \sum_k a_{nk} x_k + \sum_k a_k x_k$$

for some $t, a \in l_\infty$ and all $x \in l_A$, where the series $\sum_k a_k x_k$ and $\sum_n t_n a_{nk} x_k$ converges for $x \in l_A$. If we now set $h(x) = \sum_n t_n A_n(x)$, then it is easily seen that $h \in l'_A$. If B is an l-l matrix such that $l_A \subseteq l_B$, then $B(x) = \sum_n B_n(x)$ is a continuous linear functional on l_A, and hence

$$B(x) = \sum_n t_n A_n(x) + \sum_k a_k x_k$$

for some $t, a \in l_\infty$ and all $x \in l_A$.

2. PRELIMINARIES

An l-l method A is called perfect if l is dense in l_A in the seminorm topology, [2].

Let A be an l-l method, then we will say that a sequence $t = (t_n)$ satisfies property P if it is bounded and if $\sum_k \sum_n t_n a_{nk} x_k$ converges for every $x \in l_A$, [1].

It is shown that an l-l method A is perfect if and only if for each sequence t satisfying property P we have

$$\sum_n t_n \sum_k a_{nk} x_k = \sum_k \sum_n t_n a_{nk} x_k$$

for every $x \in l_A$, [1, Theorem A].

If A is an l-l method and if $t \in l_\infty$ and $x \in l_A$, then we write

$$t(Ax) = \sum_n t_n \sum_k a_{nk} x_k,$$

where the series is always convergent. We also write

$$(tA)x = \sum_k \sum_n t_n a_{nk} x_k,$$

whenever the series is convergent.

An l-l method A is defined to be associative if $t(Ax) = (tA)x$ for every $t \in l_\infty$ and every $x \in l_A$, [1]. Thus every associative l-l method is perfect but the converse is not generally true, [1].

Let A be an l-l method and let $x \in l_A$, then it is said that

x has AK if and only if $\sum_k x_k e_k$ converges strongly to x; x has SAK if and only if $\sum_k x_k e_k$ converges weakly to x; x has FAK if and only if $\sum_k x_k e_k$ converges weakly, [1]. Where e_k is the sequence whose k-th component is one and all others are zero.

Let $\mathcal{J}$ represent the set of all sequences in l_A which have FAK. x is associative if and only if $(tA)\,x = t\,(Ax)$ for every $t \in l_\infty$. x is perfect if and only if $(tA)\,x = t\,(Ax)$ for every sequence t which satisfies property P,

Note that the following implications are obvious:

x has AK $\Rightarrow$ x has SAK $\Rightarrow$ x has FAK.

Let us write $\sum_n A_n(x) = A\,(x)$ whenever the series $\sum_n A_n(x)$ converges.

An l-l method A is called absolutely consistent with an l-l method B if $A\,(x) = B\,(x)$ for all $x \in l_A \cap l_B$, [2].

An l-l method A is said to be replaceable if there exists an absolusolutely regular method B such that $l_A \subseteq l_B$, [2].

3. MAIN RESULTS

3.1. Replaceability, Perfectness and Associativity of l-l Methods of Summation and The Inclusion $l_A \subseteq \gamma$ for an l-l Method A.

Theorem. 3.1.1. Every absolutely regular method is replaceable but the converse is not generally true.

Proof. Let A be an absolutely regular method of summation. Now, if we set

$$f\,(x) = \sum_n A_n(x) \tag{5}$$

for every $x \in l_A$, then it is easily seen that $f \in l'_A$. By the Brown- Cowling Lemma ([2, p. 360]), there exists an *l-l* method B such that $l_A \subseteq l_B$ and

(6) $$B(x) = f(x)$$

for all $x \in l_A$. Hence

$$f(e_k) = \sum_n a_{nk} = 1 \quad (k=1,2,...)$$

and

$$f(e_k) = \sum_n b_{nk} \qquad (k=1,2,...)$$

are obtained by (5) and (6), respectively. Therefore, we have $\sum_n b_{nk} = 1$ $(k=1,2,...)$ which shows that A is replaceable.

For the converse, let us consider the following example.

Example 1. Let $A=(a_{nk})$ be a matrix with all the elements in the first and second rows are equal to one and all the other elements are zero. Since $\sum_n |a_{nk}| = 2$ for every k, A is an *l-l* method but A is not absolutely regular. Moreover it is easily shown that $l_A = \gamma$. Now, let $B = (b_{nk})$ be the matrix with all the elements in the first row are equal to one and all the other elements are zero. B is an absolutely regular method since $\sum_n |b_{nk}| = \sum_n b_{nk} = 1$ $(k=1,2,...)$. It is obvious that $l_B = \gamma$ for this method, too. Therefore A is replaceable by B.

Note that, if an absolutely regular method A is replaceable by B, then A is not need to be equivalent to B, (see [7, p. 31] for equivalent methods). In fact, let $A=(a_{nk})$ be the identity matrix and $B=(b_{nk})$ be taken as in Example 1. Then A is absolutely regular and $l_A = l$. Hence $l_A \subset l_B$ with strict inclusion. It shows that A is replaceable by B but it is not equivalent to B.

The above example shows the summability field of some *l-l* methods of summation coincides with γ. As it will seen in the following counterexample, it is not necessary to be $l_A \subseteq \gamma$.

Example 2. Let us consider the *l-l* method A defined by the matrix $A=(a_{nk})$, given in [1], as follows:

$a_{n,2n-1} = 1 (n=1,2,...)$; $a_{n,2n}=1$ $(n=1,2,...)$; $a_{nk}=0$, otherwise. Obviously, $x=(x_k)$ belongs to l_A if and only if $\sum_k |x_{2k-1}+x_{2k}| < \infty$. For example, the sequegnce $x=((-1)^{k+1})$ belongs to l_A. But $x \notin \gamma$, so $l_A \not\subseteq \gamma$.

Now, we will consider the inclusion $l_A \subseteq \gamma$ under some conditions. First of all let us give a definition and some lemmas.

The following definition is an adaptation of the definition given in [8] for c-c methods to *l-l* methods, where c is set of convergent sequences.

Definition. 3.1.2. Let A be an *l-l* method and $x \in l_A$. We say that $x=(x_k)$ satisfies properyt L if

$$(tA)\ x = \sum_k \sum_n t_n a_{nk} x_k$$

converges for every bounded sequence $t=(t_k)$.

Let $\mathcal{L}$ be the set of sequences in l_A which satisfy property L. Let us say that an *l-l* method A satisfies property L if $l_A = \mathcal{L}$.

On the other hand, $I = \{x \in c_A : \sum_k a_k x_k \text{ converges}\}$ is defined for any c-c method A where $c_A$is summability field and a_k are column limits of the matrix A, [8], and it is shown, in [4, Lemma 3], that $\mathcal{F} = \mathcal{L} \cap I$. Whereas we will obtain that $\mathcal{F} = \mathcal{L}$ for an *l-l* method in the next lemma.

Lemma. 3.1.3. For any *l-l* method A and any $x \in l_A$, x has FAK if and only if x satisfies property L. So that $\mathcal{F} = \mathcal{L}$.

Proof. let $x \in l_A$ and x has FAK. Then

$$(7) \qquad \sum_k f(e_k)\, x_k$$

converges for each $f \in l'_A$. If we write

$$(8) \qquad f(x) = \sum_n t_n A_n(x)$$

for every $x \in l_A$ and every $t \in l_\infty$, then $f \in l'_A$. By (8), we have

$$(9) \qquad f(e_k) = \sum_n t_n a_{nk} \quad (k=1,2,...).$$

Combining (9) with (7) we get the necessity.

Conversely, suppose that $x \in l_A$ and satisfies property L. If $f \in l'_A$, then f has the representation (4) for some $t, a \in l_\infty$ and for all $x \in l_A$. In particular,

$$f(e_k) = \sum_n t_n a_{nk} + a_k \quad (k=1,2,\dots).$$

Since $\sum_k a_k x_k$ converges for $x \in l_A$ and x satisfies property L, $\sum_k f(e_k)$ converges. So the proof is completed.

Using the Lemma. 3.1.3 and [1; Lemma 1, Corollary and Lemma 3] we can obtain the following lemma, immediately.

Lemma. 3.1.4. An *l-l* method A is perfect and satisfies property L if and only if it is associative.

Now, we give the inclusion theorem between l_A and γ.

Theorem. 3.1.5. Let an *l-l* method A be replaceable and satisfies property L. Then, necessarily, $l_A \subseteq \gamma$.

Proof. By the hypothesis, $l_A = \mathcal{L}$. Moreover there exists an absolutely regular method B such that $l_A \subseteq l_B$ since A is replaceable. Then $\sum_n B_n(x)$ is a continuous linear functional on l_A, and so

$$B(x) = t(Ax) + \sum_k a_k x_k \tag{10}$$

for some $t, a \in l_\infty$ and all $x \in l_A$. In particular,

$$B(e_k) = \sum_n t_n a_{nk} + a_k \quad (k=1,2,\dots). \tag{11}$$

Since B is an absolutely regular method, $B(e_k) = 1$ $(k=1,2,\dots)$. Thus, using (11), we get

$$1 = \sum_n t_n a_{nk} + a_k \quad (k=1,2,\dots). \tag{12}$$

Hence, by (12), $\sum_k x_k$ converges for every $x \in l_A$, i.e., $l_A \subseteq \gamma$ since $l_A = \mathcal{L}$ and $\sum_k a_k x_k$ converges for all $x \in l_A$.

For example, the inclusion $l_A \subset \gamma$ is strict for the identity matrix A.

Theorem. 3.1.6. Let an *l-l* method A be associative. Then A is replaceable by B if and only if

(i) $\sum_n | B_n(x) | < \infty$

(ii) $\sum_n B_n(x) = \sum_k x_k$

for evey $x \in l_A$.

Proof. Sufficiency is obvious. In this theorem, our main purpose is to show the necessity if (ii). Since the l-l method A is replaceable by B, for every $x \in l_A$, $\sum_n | B_n(x) | < \infty$ therefore $\sum_n B_n(x)$ converges. Moreover, since the matrix A is associative, it satisfies property L by Lemma. 3.1.4. Hence $l_A \subseteq \gamma$ by Theorem. 3.1.5. Then using (12), we write.

(13) $$\sum_k a_k x_k = \sum_k x_k - (tA)\ x.$$

Substituting (13) in (10), we get

$$B(x) = t(Ax) + \sum_k x_k - (tA)\ x$$

for some t, $a \in l_\infty$ and for all $x \in l_A$. Since A is associtive, we see that

$$B(x) = \sum_k x_k$$

for every $x \in l_A$which proves the theorem.

REMARK. This theorem may give the idea that l_A coincides with l since B is absolutely regular method and $B(x) = \sum_k x_k$ for every $x \in l_A$. But this is not generally true. Now we will give a counterexample, to make it clear.

Let us reconsider the matrix $A=(a_{nk})$ given in Example 1. It was shown there that A is replaceable and $l \subset \gamma = l_A$. Furthermore, it can be written that

$$\begin{aligned}(tA)x &= (t_1+t_2)\,x_2 + (t_1+t_2)\,x_2 + \ldots + (t_1+t_2)x_k + \ldots \\ &= (t_1+t_2)\sum_k x_k\end{aligned}$$

and

$$t(Ax) = t_1 \sum_k x_k + t_2 \sum_k x_k + 0 + \ldots$$

$$= (t_1+t_2)\sum_k x_k$$

for every $x \in l_A$ and for every $t \in l_\infty$, and so A is associative. Now, let $B = (b_{nk})$ be defined as follows: All the elements in the first and second rows are equal to $\frac{1}{2}$ and all the other elements are zero. Obviously, B is an absolutely regular method and $l_B = \gamma$. Hence. we have

$$B(x) = \sum_n B_n(x) = \tfrac{1}{2} \sum_k x_k + \tfrac{1}{2} \sum_k x_k + 0 + \ldots$$

$$= \sum_k x_k$$

for every $x \in l_A$. So the inclusion $l \subset l_A$ is strict.

The next corollary holds by Theorem. 3.1.1 and Theorem 3.1.6.

Corollary. 3.1.7. If an absolutely regular method A is associative, then $A(x) = \sum_k x_k$ for every $x \in l_A$.

3.2. Consistency Of Perfect *l-l* Methods Of Summation.

In this paragraph it will be shown that there exists a matrix which sums all the sequences in l_A to zero providing that $A \in (l,l)$ is perfect. This result has been obtained under a different hypothesis in [3, Theorem 3] but the proof was false. (See, M.R. vol. 52; number 3 (1976), p. 883; ≠ 6237).

Theorem. 3.2.1. If an *l-l* method A is perfect, then there exits an *l-l* method B such that $l_A \subseteq l_B$ and $B(x) = 0$ for every $x \in l_A$.

Proof. Since A is perfect, for each sequence t satisfying property P we have

(14) $$t(Ax) = (tA)x$$

for every $x \in l_A$, [1, Theorem A]. If we set

(15) $$f(x) = t(Ax) - (tA)x$$

for every $x \in l_A$ and each sequence t satisfying property P, then it is easily seen that $f \in l'_A$. Thus, by (14) and (15), we get

(16) $$f(x) = 0$$

for every $x \in l_A$. By the Brown - Cowling Lemma, ([2]), there exists an *l-l* method B such that $l_A \subseteq l_B$ and

(17) $f(x) = B(x)$

for every $x \in l_A$. So (16) and (17) give the result.

Finally, we give the following theorem dealing with absolute consistency.

Theorem. 3.2.2. Let an *l-l* method A be perfect. Then there exists an *l-l* method D which is consistent with A on l_A.

Proof. By Theorem 3.2.1 there exists an *l-l* method B such that $l_A \subseteq l_B$ and

(18) $B(x) = 0$

for every $x \in l_A$. If we set

$$d_{nk} = b_{nk} + a_{nk}$$

for all n and k, then D is an *l-l* method of summation. This relation yields the results

$$D_n(x) = B_n(x) + A_n(x), \ (n=1,2,...)$$

and

$$\sum_n |D_n(x)| \leq \sum_n |B_n(x)| + \sum_n |A_n(x)| < \infty .$$

It shows that $l_A \subseteq l_D$ and for every $x \in l_A$,

(19) $D(x) = B(x) + A(x)$.

(18) and (19) imply that

$$D(x) = A(x)$$

for every $x \in l_A$. Hence, the proof is complete.

ÖZET

Bu çalışmada; değiştirilebilir, mükemmel ve birleşebilir *l-l* toplanabilme metotlarının bazı özellikleri elde edilmiştir.

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