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Infinite Matrices and Generalized Boundedness

by

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Infinite Matrices and Generalized Boundedness

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ABSTRACT

Quite recently, Nanda has introduced the concept $\hat{l}_\infty$ of almost boundedness. In the present paper author has defined the concept L_∞^* of generalized boundedness which is related to the concept of $F\mathcal{B}$ – convergence. The concept of $F\mathcal{B}$ – convergence was introuded by Stieglitz, which is a generalization of almost convergence. Author further extends the space L_∞^* to $L_\infty^*(p)$ just as l_∞, c, c_0 and $\hat{l}_\infty$ were extended to $l_\infty(p)$, C(p), $c_0(p)$ and $\hat{l}_\infty(p)$ respectively, and characterizes certain matrices in L_∞^*.

1. INTRODUCTION

In 1948, Lorentz [1] introduced the concept of almost convergence by an application of Banach limits and characterized the space f of almost convergent sequences by means of the following property:

The sequence $x = \{x_n\}$ is almost convergent to the value f-lim x, if

$$\lim_n \frac{1}{n+1} \sum_{k=i}^{i+n} x_k = f-\lim x \qquad (\text{Uniformly } i = 0,1,\ldots)$$

This criterion can also be formulated: Let $\mathcal{B}_1 = (B_i^{(1)})$ be the sequence of matrices $B_i^{(1)} = (b_{nk}^{(1)}(i))$ with

$$b_{nk}^{(1)}(i) = \begin{cases} \dfrac{1}{n+1} & i \le k \le i+n \\ 0 & \text{otherwise.} \end{cases}$$

Thus x is almost convergent to the each value f – lim x, if

$$\lim_n (\overset{(1)}{B}_i x)_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} \overset{(1)}{b}_{nk} (i)\, x_k = f - \lim x$$

(Uniformly $i = 0,1, \ldots\ldots$)

Stieglitz [9] further generalized this concept by means of a given matrix sequence $\mathcal{B} = (B_i)$ with $B_i = (b_{nk}(i))$, x of the space l_∞ of bounded sequences is $F_{\mathcal{B}}$ -convergent to the value $\mathrm{Lim}_{\mathcal{B}}\, x$, if

$$\lim_{n\to\infty} (B_i\, x)_n = \lim_{n\to\infty} \sum_{k=0}^{\infty} b_{nk}(i)\, x_k = \mathrm{Lim}_{\mathcal{B}}\, x$$

(Uniformly $i = 0,1, \ldots\ldots.$)

Recently, Nanda [7] defined the concept $\hat{l}_\infty$ of almost boundedness in the following manner:

$$\hat{l}_\infty = \{ a \in s : \sup_{n,i} | \Phi_{n,i}(a) | < \infty \}$$

where

$$x_n = a_0 + a_1 + \ldots\ldots\ldots\ldots\ldots + a_n$$

$$\Phi_{n,i}(a) = t_{n,i}(x) - t_{n-1,i}(x)$$

and

$$t_{n,i}(x) = \frac{1}{n+1} \sum_{k=i}^{i+n} x_k$$

$$= \sum_{k=0}^{\infty} \overset{(1)}{b}_{nk} (i)\, x_k$$

Therefore $\hat{l}_\infty$ can also be formulated as:

$$\hat{l}_\infty = \{ x \in s: \sup_{n,i} | (\overset{(1)}{B}_i x)_n - (\overset{(1)}{B}_i x)_{n-1} | < \infty \}$$

We generalize the space $\hat{l}_\infty$ by means of a given matrix sequence $\mathcal{B} = (B_i)$ with $B_i = (b_{nk}(i))$ as follows:

$$L^*_\infty = \{ x \in s: \sup_{n,i} | \Psi_{n,i}(x) | < \infty \}$$

where

$$\Psi_{n,i}(x) = (B_i\, x)_n - (B_i\, x)_{n-1}$$

$$= \sum_k (b_{nk}(i) - b_{n-1,k}(i))\, x_k.$$

We further extend L^*_∞ to $L^*_\infty(p)$ just as l_∞, l, c and $\hat{l}_\infty$ were extended to $l_\infty(p)$, $l(p)$, c (p) and $\hat{l}_\infty(p)$ respectively (see Maddox [4], Simons [8] and Nanda [7]).

If p_n is a real number such that $p_n > 0$ and $\sup p_n < \infty$, we write

$$L^*_\infty(p) = \{ x \in s: \sup_{n,i} | \Psi_{n,i}(x) |^{p_n} < \infty \}$$

If $p_n = p \ \forall\ n$, then $L^*_\infty(p) = L^*_\infty$.

The space L^*_∞ depends on the fixed chosen matrix $\mathcal{B} = (B_i)$. In case $\mathcal{B}_o = (I)$ (unit matrix) it is equal to l_∞. In case

$\mathcal{B}_1 = (b^{(1)}_{nk}(i))$ it is same as $\hat{l}_\infty$.

2. Some Topological Results.

Theorem 2.1. If $\inf p_n > o$, then $L^*_\infty(p)$ is a complete linear topological space over the complex field $\mathbb{C}$ paranormed by g defined by

$$g(x) = \sup_{n,i} | \Psi_{n,i}(x) |^{p_n/M} \quad (\forall\ x \in L^*_\infty(p));$$

where $M = \max(1, \sup p_n)$.

Proof. Since $p_n/M \leq 1$, we have (See Maddox [6], p. 31)

$$(1)\ | \Psi_{n,i}(x + y) |^{p_n/M} \leq | \Psi_{n,i}(x) |^{p_n/M} + | \Psi_{n,i}(y) |^{p_n/M}, (\forall n,i)$$

and $\forall\ \lambda \in \mathbb{C}$ (See Maddox [6], p. 346),

$$(2)\ | \lambda |^{p_n/M} \leq \max(1, | \lambda |).$$

Therefore linearity follows from (1) and (2).

A linear topological space X is called a paranormed space if there exists a sub additive function $g: X \to R^+$ such that $g(0) = 0$, $g(x) = g(-x)$ and the multiplication is continuous, that is,

$\lambda_n \to \lambda$ and $g(x_n - x) \to 0$ imply that

$g(\lambda_n x_n - \lambda x) \to 0$ for all $\lambda \in \mathbb{C}$ and $x \in X$.

It is easy to see that $g(0) = o$ and $g(x) = g(-x)$ for every $x \in L^*_\infty(p)$. The subadditivity of g follows from (1) by taking supremum with respect to n and i. It follows from (2) that for $\lambda \in \mathbb{C}$ and $x \in L^*_\infty(p)$

$$g(\lambda x) \leq \max(1, |\lambda|)\, g(x)$$

Therefore $\lambda \to 0, x \to o, \Longrightarrow \lambda x \to 0$ and if λ is fixed, $x \to 0 \Longrightarrow \lambda x \to 0$. Let $\inf p_n = \beta > 0$. Then we have

$$g(\lambda x) \leq \max(|\lambda|, |\lambda|^{\beta/M})\, g(x).$$

Hence for fixed x, $\lambda \to 0 \Longrightarrow \lambda x \to 0$.

Let $\{x^j\}$ be a Cauchy sequence in $L^*_\infty(p)$. Then $\{x_k{}^j\}$ for each k, is a Cauchy sequence in $\mathbb{C}$ and hence $x^j{}_k \to x_k$ for each k. Put $x = \{x_k\}$. Now it can be easily seen that $x \in L^*_\infty(p)$ and $g(x^j - x) \to 0$.

This terminates the proof.

Proposition 1. Let $\mathcal{B} = (B_i)$ be a family of matrices with $N(\mathcal{B}) < \infty$. Then $l^*_\infty \subset L^*_\infty$

Proof. Let $x \in l_\infty$. We have

$$\sup_{n,i} |\Psi_{n,i}(x)| \leq N(\mathcal{B})\, ||x||_\infty < \infty$$

Therefore $x \in L^*_\infty$ and hence $l_\infty \subset L^*_\infty$.

For $r > o$, a nonempty subset U of a linear space is said to be absolutely r-convex if $x, y \in U$ and

$|\lambda|^r + |\mu|^r \leq 1$ together imply that $\lambda x + \mu y \in U$.

A linear topological space X is said to be r-convex (See Maddox and Roles [5]) if every neighbourhood of $0 \in X$ contains as absolutely r-convex neighbourhood of $0 \in X$. We have.

Proposition 2. $L^*_\infty(p)$ is 1-convex

Proof. If $o < \delta < 1$, then

$$U = \{x : g(x) \leq \delta\}$$

is an absolutely 1-convex set, for let $x, y \in U$ and $|\lambda| + |\mu| \leq 1$, then

$$g(\lambda x + \mu y) \leq (|\lambda| + \mu|)^{P_n/M} . \quad \delta \leq \delta.$$

This completes our proof.

Theoıem 2.2. Let $o < p_n \leq q_n$, then L^*_∞ (q) is a closed subspace of L^*_∞ (p).

Proof. Let $x \in L^*_\infty$ (q). Then there exists a constant $K > 1$ such that $\forall$ n,i.

$$|\Psi_{n,i}(x)|^{q_n/M} \leq K.$$

implies that

$$|\Psi_{n,i}(x)|^{p_n/M} \leq K.$$

Therefore $x \in L^*_\infty$ (p). Now suppose that $x^j \in L^*_\infty$ (q) and $x^j \to x \in L^*_\infty$ (p). Then for every $0 < \in < 1$, there exists an integer N such that for every n,i

$$|\Psi_{n,i}(x^j - x)|^{p_n/M} < \in \qquad (\forall j > N)$$

implies that

$$\| \Psi_{n,i}(x^j - x)|^{q_n/M} < \quad |\Psi_{n,i}(x^j - x)|^{p_n/M} < \in$$

Therefore $x \in L^*_\infty$ (q). And hence the proof is complete.

3. In this section we consider matrix transformations between some class of sequences. We write

$$\Psi_{n,i}(Ax) = (B_i(Ax))_n - (B_i(Ax))_{n-1}$$

$$= \sum_m \sum_k (b_{nk}(i) - b_{n-1,k}(i))\, a_{km}\, x_m$$

$$= \sum_m \alpha(i, m, n)\, x_m$$

wheıe

$$\alpha(i, m, n) = \sum_k (b_{nk}(i) - b_{n-1,k}(i))\, a_{km}.$$

Theorem 3.1. $A \in (l^*_\infty, L^*_\infty)$ if and only if

$$\sup_{n,i} \sum_m |\alpha(i, m, n)| < \infty$$

Proof. Necessity. Let $A \in (l_\infty, L^*_\infty)$. Put

$$f_i(x) = \sup_n |\Psi_{n,i}(Ax)|.$$

Now $\{f_i\}$ is a sequence of continuous seminorms on l_∞ such that $\sup_i f_i(x)$ is finite. Therefore by Banach-Steinhaus theorem (See Maddox [6], p. 114) there exists a constant K such that for every i, $x \in l_\infty$

$$f_i(x) \leq K \|x\|_\infty$$

Now by putting $x = \operatorname{sgn} \alpha(i, m, n)$ in this above inequality necessity follows immediately.

Sufficiency. Suppose that our condition holds and $x \in l_\infty$. Then

$$\sup_{n,i} |\Psi_{n,i}(Ax)| \leq \sup_{n,i} \sum_m |\alpha(i, m, n) x_m|$$

$$\leq \|x\|_\infty \sup_{n,i} \sum_m |\alpha(i, m, n)|$$

This completes the proof.

Theorem 3.2. $A \in (l_\infty(p), L^*_\infty)$ if and only if for every integer $N > 1$

$$(3.2.1) \quad \sup_{n,i} \sum_m |\alpha(i, m, n)| N^{1/p_m} < \infty.$$

Proof. Necessity. Suppose that $A \in (l_\infty(p), L^*_\infty)$, and let there exists an integer $N > 1$ such that (3.2.1) does not hold. Therefore by Theorem (3.1), the matrix

$C = (C_{nk}) = (a_{nk} N^{1/p_k}) \notin (l_\infty, L^*_\infty)$, i.e., there exists $x \in l_\infty$ such that $Cx \notin L^*_\infty$. Now $y = \{y_k\} = \{x_k N^{1/p_k}\} \in l_\infty(p)$. But we have $Ay = Cx \notin L^*_\infty$, which is a contradiction to the fact that $A \in (l_\infty(p), L^*_\infty)$.

Sufficiency. Let us suppose that our condition (3.2.1) holds. If we take $N > \max(1, \sup_m |x_m|^{p_m})$, then for every n and i

$$|\Psi_{n,i}(Ax)| \leq \sum_m |\alpha(i, m, n)|^{1/p_m} N^{1/p_m}$$

Therefore taking supremum over n and i, sufficiency follows immediately. Hence this completes the proof.

Theorem 3.3. $A \in (l\,(p), L^*_\infty)$ if and only if

(i) There exists and integer $N > 1$ such that

$$\sup_{n,i} \sum_m |\alpha(i, m, n)|^{q_m} N^{-q_m} < \infty, \; (1 < p_m < \infty, p_m^{-1} + q_m^{-1} = 1)$$

(ii) $\sup_{n,i,m} |\alpha(i,m,n)|^{p_m} < \infty \; (0 < p_m \leq 1)$.

Proof. Necessity. Suppose that $A \in (\ (p), L^*_\infty)$ and put

$$T_{n,i}(x) = \Psi_{n,i}(Ax)$$

and

$$f_i(x) = \sup_n |\Psi_{n,i}(Ax)|.$$

we see that $\{T_{n,i}\}$ being a sequence of continuous real functions on $l\,(p)$ for each n, and $\{f_i\}$ is also a sequence of continuous real functions on lp and $\sup_i f_i(x) < \infty$. then the result follows immediately by uniform boundedness principle (see Lascarides and Maddox [2], Theorem 1). This proves the necessity.

Sufficiency. Here we only consider the case $1 < p_m < \infty$. Suppose that the conditions (i) and (ii) hold and $x \in l\,(p)$. Since we know the following inequality (See Lascarides and Maddox [2], p. 100): If $x, y \in$ (∮ and $N > 0$ then

$$|xy| \leq N(|x|^{q_m} N^{-q_m} + |y|^{p_m})$$

wbere

$$1 < p_m < \infty \text{ and } p_m^{-1} + q_m^{-1} = 1.$$

Hence

$$|\Psi_{n,i}(Ax)| \leq \sum_m N(|\alpha(i, m, n)|^{q_m} N^{-q_m} + |x_m|^{p_m}).$$

taking the supremum over n and i, we observe $A \in (l\,(p), L^*_\infty)$. And hence our theorem is proved.

Theorem 3.4. $A \in (C_0(p), L^*_\infty(p))$ if and only if there exists an integer $N > 1$ such that

$$\sup_{n,i} \left\{ \sum_m |\alpha(i,m,n)| N^{-1/p_m} \right\}^{p_n} < \infty.$$

Proof. Necessity. Suppose that $A \in (C_o(p), L^*_\infty(p))$ and $x \in C_o(p)$. Put

$$T_{n,i}(x) = |\Psi_{n,i}(Ax)|^{p_n}$$

and

$$T_i(x) = \sup_n T_{n,i}(x).$$

Since $\{T_{n,i}\}$ and $\{T_i\}$ are the sequences of continuous real function on $c_o(p)$ and $\sup_i T_i(x)$ is finite. Therefore by uniform boundedness necessity is proved.

Sufficiency. There is no need to prove this part. Since it can be easily obtained by an anaylsis similar to Lascarides [3], Theorem 10. Hence the proof is complete.

Finally, the author is grateful to Dr. Z. U. Ahmad for suggestions and guidance.

ÖZET

Yakın bir zamanda Nanda; [7] de hemen hemen sınırlılık, l^*_∞, kavramını tanımlamıştı. Bu çalışmamızda, F^*_∞ – yakınsaklıkla ilişkili L^*_∞ genelleştirilmiş sınırlılık kavramını tanımladık. Hemen hemen yakınsaklığın bir genelleştirilmesi olan F^*_∞ – yakınsaklığı Stieglitz tarafından verilmiştir. Ayrıca çalışmamızda; l^*_∞, c, c_o uzaylarının sırasıyle l^*_∞ (p), c (p) ve c_o (p) ye genişletilmesi gibi, L^*_∞ uzayını da L^*_∞ (p) ye genişleterek, L^*_∞ üzerinde bazı matrisleri karekterize ettik.

REFERENCES

[1]. **G.G. Lorentz,** A contribution to the theory of divergent sequences, Acta Math. 80 (1948), 167 – 190.

[2]. **G.G. Lascarides and I.J. Maddox,** Matrix transformations between some classes of sequences, Proc. Camb. Philos. Soc., 68 (1970), 199 – 204.

[3]. **G.G. Lascarides,** A study of certain sequences spaces of Maddox and a generalization of a theorem of Iyer, Pacific Jour. Math., 38 (2) (1971), 487 – 500.

[4]. **I.J. Maddox,** Spaces of strongly summable sequences, Quart. Journ. Math., 2 (18) (1967) 345 – 365.

[5]. **I.J. Maddox and J.W. Roles,** Absolute Convexity in certain topological linear spaces, Proc. Camb. Philo. Soc., 66 (1969), 541 – 545.

[6]. **I.J. Maddox,** Elements of functional analysis, Camb. Univ. Press, (1970).

[7]. **S. Nanda,** Some sequences spaces and almost boundedness, MATEMATNYK BECHNK, 13 (28) (1976), 179 – 186.

[8]. **S. Simons,** The sequences spaces l (p_ν) and m (p_ν), Proc. London. Math. Soc., 3 (15) (1965), 422 – 436.

[9]. **M. Stieglitz,** Eine Verallgemeinerung des Begriffs des Fastkonvergenz, Math. Japon, 18 (1973), 53 – 70.

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