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Matrix Transformations of Some Generalized Sequence spaces into semiperiodic Sequence Space

by

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DEDICATION TO ATATÜRK'S CENTENNIAL

Holding the torch that was lift by Atatürk in the hope of advancing our. Country to a modern level of civilization, we celebrate the one hundredth anniversary of his birth. We know that we can only achieve this level in the fields of science and technology that are the wealth of humanity by being productive and creative. As we thus proceed, we are conscious that, in the words of Atatürk, "the truest guide" is knowledge and science.

As members of the Faculty of Science at the University of Ankara we are making every effort to carry out scientific research, as well as to educate and train technicians, scientists, and graduates at every level. As long as we keep in our minds what Atatürk created for his Country, we can never be satisfied with what we have been able to achieve. Yet, the longing for truth, beauty, and a sense of responsibility toward our fellow human beings that he kindled within us gives us strength to strive for even more basic and meaningful service in the future.

From this year forward, we wish and aspire toward surpassing our past efforts, and with each coming year, to serve in greater measure the field of universal science and our own nation.

Matrix Transformations of Some Generalized Sequence Spaces Into Semiperiodic Sequence Space

S.M. SIRAJUDEEN*

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ABSTRACT

Necessary and sufficient conditions have been established on an infinite matrix $A = (a_{nk})$ to transform the generalized sequence spaces $l(p)$, $c_0(p)$ and $l_\infty(p)$ into, q, the space of semiperiodic sequences. The special case when $p_k = 1/k$ gives $c_0(p)$ and $l_\infty(p)$ as Γ and Γ^*, the spaces introduced by V. Ganapathy Iyer [2].

INTRODUCTION

Inclusion theorems on an infinite matrix (a_{nk}) to transform the normable space c_0, c, l and the non-normable spaces Γ and Γ^* into q, the space of semiperiodic sequences have been established by K. Chandresekhara Rao in [1] and that to transform l_p $(p > 1)$ into q has been established by D. Somasundaram and S.M. Sirajudeen in [11]. The generalized sequence spaces $l(p)$, l_∞ (p), $c_0(p)$ and $c(p)$ have been introduced by I.J. Maddox in [5]. Hence it is worth to find out necessary and sufficient conditions on an infinite matrix (a_{nk}) to transform the generalized sequence spaces into q. The object of this paper is to characterize matrix transformations of $l(p)$, $c_0(p)$ and $l_\infty(p)$ into q and many of the results in [1] and [11] have become particular cases of our results.

In § 2 the required sequence spaces have been defined and some known results have been quoted as lemmas. In § 3, we investigate the necessary and sufficient conditions on an infinite matrix in order that it should transform

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$l(p)$	into	q	(Theorem 1)
$c_o(p)$	into	q	(Theorem 2)
$l_\infty(p)$	into	q	(Theorem 3)

2. Writing an infinite sequence $(x_1, x_2, \ldots, x_k, \ldots)$ by $x = (x_k)$, $\sum_{k=1}^{\infty} a_{nk}$ by $\Sigma\, a_{nk}$
and $p = (p_k)$ as a bounded sequence of strictly positive real numbers, let us define the required sequence spaces as follows (see [1], [2], [3], [4], [5], [6], [7], [8], [9] and [11]):

$l(p) = \{x = (x_k): \Sigma\, |x_k|^{p_k} < \infty\}$. $l(p)$ is linear and complete under the paranorm $g(x) = (\Sigma |x_k|^{p_k})^{1/H}$ where $H = \max(1, \sup_k p_k)$.

$l_\infty(p) = \{x = (x_k): \sup_k |x_k|^{p_k} < \infty\}$

$c_o(p) = \{x = (x_k): |x_k|^{p_k} \to 0 \text{ as } k \to \infty\}$. $c_o(p)$ is a proper subset of m the space of bounded sequences.

$q = \{x = (x_k): (x_k) \text{ is semiperiodic}\}$. A sequence $x = (x_k)$ is said to be semiperiodic if to each $\varepsilon > 0$, there exists a positive integer i such that $|x_k - x_{k+ri}| < \varepsilon$ for all r and k. The sequence q is a separable subspace of m of bounded sequences.

When $p_k = 1$ for all k, we write $l(p) = l$, $l_\infty(p) = m$ and $c_o(p) = c_o$. When $p_k = 1/k$, we write $l_\infty(p) = \Gamma^*$ and $c_o(p) = \Gamma$; and when $p_k = p > 1$ for all k, we write $l(p) = l_p$.

If X and Y are any two sequence spaces, let (X,Y) denote the class of all matrices $A = (a_{nk})$; n, k=1,2,... that transforms $x = (x_k) \in X$ into $Ax = y = (y_n) \in Y$ defined by

$$y_n = \Sigma\, a_{nk} x_k; \qquad n = 1, 2, \ldots.$$

Now let us quote some known results as the following
LEMMA A (Theorem 1 [4]). $A \in (l(p), m)$ if and only if

(i) $\sup_{n,k} |a_{nk}|^{p_k} < \infty$ when $0 < p_k \le 1$

and

(ii) there exists an integer $M > 1$ such that

$$\sup_n \Sigma \, |a_{nk}|^{q_k} M^{-q_k} < \infty \text{ when } 1 < p_k \, \lambda \leq \operatorname{Sup} p_k < \infty$$

where $p_k^{-1} + q_k^{-1} = 1$.

LEMMA B (Theorem 10 and 11 [3]). $A \in (c_0(p), l_\infty(p))$ if and only if there exists an absolute constant $M > 1$ such that

$$\sup_n (\Sigma \, |a_{nk}| \, M^{-1/p_k p_n}) < \infty$$

which is equivalent to

$$\sup_{n,k} |a_{nk}|^{1/(r_k + s_n)} < \infty$$

when Q is the set of all $p = (p_k)$ for which there exists $N > 1$ such that $\Sigma \, N^{-r_k} < \infty$ where $r_k = 1/p_k$ and $s_n = 1/p_n$.

LEMMA C (Theorem 3 [4] and Corollary 2 of Theorem 2 [10]). $A \in (l_\infty(p), m)$ if and only if

$$\sup_n \Sigma \, |a_{nk}| \, M^{1/p_k} < \infty \text{ for every integer } M > 1.$$

3. Now we shall establish the three theorems from which we can deduce as corollaries some of the known results as particular cases. Throughout this paperλ,let ϵ and e_k denote respectively the sequences (1, 1, 1,...) and (0,0,..., 0,1,0,...) the 1 in the kth place.

THEOREM 1. $A \in (l(p), q)$ if and only if

(1.1) each column of the matrix $A = (a_{nk})$ belongs to q and

(1.2) $\sup_{n,k} |a_{nk}|^{p_k} < \infty$ when $0 < p_k \leq 1$

or

there exists an integer $M > 1$ such that

$\sup_n \Sigma \, |a_{nk}|^{q_k} M^{-q_k} < \infty$ when $1 < p_k \, \lambda \leq \sup p_k < \infty$ where $p_k^{-1} + q_k^{-1} = 1$.

PROOF: Let $A \in (l(p), q)$. Since $e_k \in l(p)$, the necessity of (1.1) is trivial.

Since $q \subset m$, the necessity of (1.2) follows from Lemma A.
Conversely let (1.1) and (1.2) hold and $(x_k) \in l(p)$. Then

(1.3) $|a_{nk}|^{p_k} \leq M$ independent of n and k when $0 < p_k \leq 1$ or $\Sigma\ |a_{nk}|^{q_k} M^{-q_k} < \infty$ when $1 < p^k \leq \sup p_k < \infty$ where $p_k^{-1} + q_k^{-1} = 1$.

Since $(x_k) \in l(p) \subset c_0(p) \subset m$, given an $\varepsilon > 0$, there exists a $P \geq 1$ and L independent of k such that

$$(1.4) \qquad \sum_{k=P+1}^{\infty} |x_k|^{p_k} < \frac{\varepsilon}{8M}$$

and

$$(1.5) \qquad |x_k| \leq L$$

When P is fixed, by (1.1) for a given $\varepsilon > 0$ and for all n and r, there exists a i_k such that $|a_{nk} - a_{n+ri,k}| < \frac{\varepsilon}{2PL}$ so tahat

$$(1.6) \qquad \sum_{k=1}^{P} |a_{nk} - a_{n+ri,k}| < \frac{\varepsilon}{2L}$$ where i is the least common multiple of i_k, $k = 1,2,..,P$

Now

$$(1.7) \qquad |y_n - y_{n+ri}| \leq S_1 + S_2 \text{ where } S_1 = \sum_{k=1}^{P} |a_{nk} - a_{n+ri,k}|\ |x_k|$$

and $$S_2 = \sum_{k=P+1}^{\infty} |a_{nk} - a_{n+ri,k}|\ |x_k|$$

Case (i): When $0 < p_k \leq 1$, we have if $g(x) = \Sigma\ |x_k|^{p_k} < 1/M$, since $M^{\frac{1}{p_k}} |x_k| < 1$, using (1.3) we have

$$S_2 \leq 2 \sum_{k=P+1}^{\infty} M^{\frac{1}{p_k}} |x_k| \leq 2 \sum_{k=P+1}^{\infty} M\ |x_k|^{p_k} < \frac{\varepsilon}{2} \text{ using (1.4)}$$

and $$S_1 < \frac{\varepsilon}{2} \text{ using (1.5) and (1.6)}$$

Hence (1.7) gives $|y_n - y_{n+ri}| < \varepsilon$ so that $(y_n) \in q$.

Case (ii): When $1 < p_k \leq \sup p_k < \infty$ and $p_k^{-1} + q_k^{-1} = 1$, from the inequality $|ax| \leq |a|^s + |x|^p$ for any two complex numbers a, x where $p > 1$ and $p^{-1} + s^{-1} = 1$, we get

(1.8) $|ax| \leq B(|a|^s B^{-s} + |x|^p); B > 0$

Hence $S_1 < \frac{\varepsilon}{2}$ using (1.5) and (1.6)

$$S_2 \leq M \left(\sum_{k=p+1}^{\infty} (|a_{nk}|^{q_k} + |a_{n+ri,k}|^{q_k}) M^{-q_k} + 2 \sum_{k=p+1}^{\infty} |x_k|^{p_k} \right)$$

using (1.8)

$< \frac{\varepsilon}{2}$ using (1.3) an (1.4)

Therefore (1.7) gives $|y_n - y_{n+ri}| < \varepsilon$ so that $(y_n) \in q$.

COROLLARY 1 (Theorem 3 [1]). $A \in (l, q)$ if and only if

(i) each column of the matrix $A = (a_{nk})$ belongs to q and

(ii) $|a_{nk}| \leq M$ independently of n and k

PROOF: Take $p_k = 1$ for all k.

COROLLARY 2 (Theorem 4 [11]). $A \in (l_p, q)$ if and only if

(i) each column of the matrix $A = (a_{nk})$ belongs to q and

(ii) $\sup_n \Sigma |a_{nk}|^s < \infty$ when $p > 1$ where $p^{-1} + s^{-1} = 1$.

PROOF: Take $p_k = p$ for all k so that $q_k = s$ for all k.

THEOREM 2. $A \in (c_o(p), q)$ if and only if

(2.1) each column of the matrix $A = (a_{nk})$ belongs to q and

(2.2) there exists an absolute constant $M > 1$ such that

$$\sup_n \Sigma |a_{nk}| M^{\frac{-1}{p_k}} < \infty$$

which is equivalent to

$$\sup_{n,k} |a_{nk}^{\frac{1}{r_k}}| < \infty$$

when Q is the set of all $p = (p_k)$ for which there exists $N > 1$ such that $\Sigma N^{-r_k} < \infty$ where $r_k = p_k^{-1}$.

PROOF: Let $A \in (c_o(p), q)$. Since $e_k \in c_o(p)$, the necessity of (2.1) is obvious.

Since $q \subset m$, the necessity of (2.2) follows from Lemma B when $P_n = 1$ for all n.

Conversely let (2.1) and (2.2) hold and $(x_k) \in c_o(p)$. Then given an $\varepsilon > 0$, there exists a $P \geq 1$ and an integer $M > 1$ such that

(2.3) $\quad |x_k| \leq M^{-1/p_k}$ for all $k \geq P$ and

(2.4) $\quad \sum_{k=P+1}^{\infty} |a_{nk}| M^{\frac{-1}{p_k}} < \frac{\varepsilon}{4}$

Since $(x_k) \in c_o(p) \subset m$, there exists a L independent of k such that

(2.5) $\quad |x_k| \leq L$

When P is fixed, by (2.1), for $\varepsilon > 0$ and for all n and r, there exists a i_k such that $|a_{nk} - a_{n+ri_k, k}| < \frac{\varepsilon}{2LP}$ so that

(2.6) $\quad \sum_{k=1}^{P} |a_{nk} - a_{n+ri,k}| < \frac{\varepsilon}{2L}$ where i is least common multiple of i_k, $k=1,2,..,P$

Then as in Theorem 1.

(2.7) $\quad |y_n - y_{n+ri}| \leq S_1 + S_2$

Using (2.5) and (2.6), we get $S_1 < \frac{\varepsilon}{2}$,

and $\quad S_2 \leq \sum_{k=P+1}^{\infty} |a_{nk} - a_{n+ri,k}| M^{-1/p_k}$ using (2.3)

$< \frac{\varepsilon}{2}$ using (2.4)

Hence (2.7) gives $|y_n - y_{n+ri}| < \varepsilon$ so that $(y_n) \in q$.

COROLLARY (Theorem 1 [1]). $A \in (c_o, q)$ if and only if

(i) each column of the matrix $A = (a_{nk})$ belongs to q and

(ii) $\Sigma |a_{nk}| \leq M$ independet of n

PROOF: Take $p_k = 1$ for all k.

COROLLARY 2 (Theorem 4 [1]). $A \in (\Gamma, q)$ if and only if

(i) each column of the matrix $A = (a_{nk})$ belonge to q and

(ii) $|a_{nk}|^{1/k} \leq D$ independent of n and k.

PROOF: Take $p_k = 1/k$.

THEOREM 3. $A \in (l_\infty(p), q)$ if and only if

(3.1) each column of the matrix $A = (a_{nk})$ belongs to q and

(3.2) $\sup_n \Sigma |a_{nk}| M^{1/p_k} < \infty$ for eevry integer $M > 1$

PROOF: Let $A \in (l_\infty(p), q)$. Since $e_k \in l_\infty(p)$, (3.1) is necessary.

Since $q \subset m$, the necessity of (3.2) follows from Lemma C.

Conversely let (3.1) and (3.2) hold and $(x_k) \in l_\infty(p)$. Choose an integer $M > \max(1, \sup |x_k|^{p_k})$. Then there exists a $P \geq 1$ such that

(3.3) $|x_k| \leq M^{\frac{1}{p_k}} \leq M^{1/h}$ for all k where $h = \inf p_k$ and

(3.4) $\sum_{k=P+1}^{\infty} |a_{nk}| M^{\frac{1}{p_k}} < \frac{\varepsilon}{4}$

When P is fixed, by (3.1), for $\varepsilon > 0$ and for all n and r, there exists a i_k such that $|a_{nk} - a_{n+ri_k,k}| < \frac{\varepsilon}{2PM^{1/h}}$ so that

(3.5) $\sum_{k=1}^{P} |a_{nk} - a_{n+ri,k}| < \frac{\varepsilon}{2M^{1/h}}$ where i is the least common multiple of i_k, $k = 1,2,..,P$

Then as in Theorem 1,

(3.6) $|y_n - y_{n+ri}| \leq S_1 + S_2$

Using (3.3) and (3.5), we get $S_1 < \frac{\varepsilon}{2}$;

Using (3.3), $S_2 \leq \sum_{k=P+1}^{\infty} |a_{nk} - a_{n+ri,k}| M^{\frac{1}{p_k}}$

$$< \frac{\varepsilon}{2} \text{ using } (3.4)$$

Hence (3.6) gives $|y_n - y_{n+r}| < \varepsilon$ so that $(y_n) \in q$.

COROLLARY 1. $A \in (m, q)$ if and only if

(i) each column of the matrix $A = (a_{nk})$ belongs to q and

(ii) $\sup_n \Sigma |a_{nk}| < \infty$

PROOF: Take $p_k = 1$ for all k.

COROLLARY 2. $A \in (\Gamma^*, q)$ if and only if

(C_1) each column of the matrix $A = (a_{nk})$ belongs to q and

(C_2) $\sup_n \Sigma |a_{nk}| M^k < \infty$ for every integer $M > 1$

PROOF: Take $p_k = 1/k$.

This can also be written in another form (Theorem 5 [1]) as $A \in (\Gamma^*, q)$ if and only if

(C_3) each column of the matrix $A = (a_{nk})$ belongs to q and

(C_4) the sequence $\{f_n(z)\}$ of integral functions is uniformly bounded on every compact set (of the complex plane) where $f_n(z) = \Sigma a_{nk} z^k$; $n=1,2,\ldots$.

PROOF: (C_1) and (C_3) are the same.

To prove (C_1) and (C_4) are equivalent.

Let (C_2) hold. Then clearyl (C_2) implies (C_4).

Conversely let $f_n(z) = \Sigma a_{nk} z^k$; $n = 1,2,\ldots$ be uniformly bounded on the compact set $|z| \leq R$. Then by (C_4), there is aconstant $P(R)$ such that $|f_n(z)| \leq P(R)$ for all z such that $|z| \leq R$. By Cauchy's inequality $|a_{nk}| \leq P(R)/R^k$. Taking $R = 2M$, we have

$$\Sigma |a_{nk}| M^k \leq \Sigma \frac{P(R).M^k}{R^k} \leq P(2M) \Sigma \frac{1}{2^k} \text{ for all n. Hence we get } (C_2).$$

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ÖZET

Bu çalışmada bir $A = (a_{nk})$ sonsuz matrisinin $l(p)$, $c_0(p)$ ve $l_\infty(p)$ gelleştirilmiş dizi uzaylarını yarı-periyodik dizi uzayı q'nun içine dönüştürmesi için sağlaması gereken gerek ve yeter koşullar bulunmuştur. $c_0(p)$ ve $l(p)$ uzayları $p_k = 1/k$ alınmas halinde V. Ganapathy I yer 2 tarafından tanımlanan Γ ve Γ* uzaylarını vermektedir.

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