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The Sheaf Of The Fundamental Groups*

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The Sheaf Of The Fundamental Groups*

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SUMMARY

In this paper, we consider both homotopy and sheaf theory and construct an algebraic sheaf by means of the fundamental group. Finally, we give some algebraic topogical characterizations

1. The sheaf of the fundamental groups of a topological space.

First recall the following definition.

Definition 1.1. Let X, S be two topological spaces, and π: S $\rightarrow$ X be a locally topological map. Then the pair (S, π) or shortly S is called a sheaf over X.

Let X be a locally arcwise connected topological space. Then, X has a basis of arcwise connected open sets. On the other hand, if $W \subset X$ is any arcwise connected open set and x, y are points in W, then the fundamental groups $\pi_1(X,x)$ and $\pi_1(X, y)$ are isomorphic. From now on, X will be considered as a locally arcwise connected topological space [2].

Let us denote by H the disjoint union of the fundamental groups obtained for each $x \in X$, i. e., $H = \bigvee_{x \in X} \pi_1(X, x)$. Thus H is a set over X. Let us now define a map φ: $H \rightarrow X$ as $\sigma \in H \Rightarrow \exists\, x \in X \ni \sigma \in \pi_1(X, x) \Rightarrow \sigma = [\alpha]_x \Rightarrow \varphi(\sigma) = x$. Now, if $x \in X$ is an arbitrarily fixed point, then let us denote by W=W(x) the arcwise connected open neighborhood of x in X. If $[\alpha]_x$ is a homotopy class in $\pi_1(X, x)$ and y is any point in W, then there exists a homotopy class $[\beta]_y$ in π_1 (X, y) which uniquely corresponds to $[\alpha]_x$, since $\pi_1(X, x)$ is isomorphic

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to $\pi_1(X, y)$. Therefore for any $W=W(x)$ we can define a mapping s: $W \to H$ as follows:

If $[\alpha]_x$ is an arbitrarily fixed homotopy class in $\pi_1(X,x)$, then for each $y \in W$, $s(y) = [(\gamma^{-1}\alpha)\gamma] = [\beta]_y$, where γ is an arbitrarily fixed arc in W with initial point x and terminal point y. Clearly s is well-defined and

1. $s(x) = [\alpha]_x \in \pi_1(X, x)$.

2. $\varphi \circ s = 1_W$.

Let $T = \{s(W): W=W(x) \subset X\} \cup \{H\}$. Then, T is a basis for a topology on H whose open sets are arbitrary unions of elements of T. In fact, for any two elements $s_1(W_1), s_2(W_2) \in T$

1- If the intersection $s_1(W_1) \cap s_2(W_2) \neq \varnothing$, then there exists at least one point y in $W_1 \cap W_2$. Hence $W_1 = W_2$ and $s_1(W_1) = s_2(W_2)$. Thus, $s_1(W_1) \cap s_2(W_2) = s_1(W_1) \in T$.

2- If the intersection $s_1(W_1) \cap s_2(W_2) = \varnothing$, then $s_1(W_1) \cap s_2(W_2) \in T$, since $\varnothing \in T$.

Let us now show that φ is locally topological. If $\sigma = [\beta]_y \in H$, $y \in X$, then $\varphi(\sigma) = \varphi([\beta]_y) = y$. Hence, there exists a mapping $s: W \to H \ni s(y) = \sigma$, $y \in W = W(x)$. Now, let us assume that $U(\sigma) = s(W)$ and $\varphi \mid U = \varphi^*$.

1. The mapping $\varphi^* = \varphi \mid U = s(W): U \to W$ is injective. Because for any $\sigma_1, \sigma_2 \in s(W)$ there are arcs γ_1, γ_2 in W with initial point x, terminal points y_1, y_2, respectively such that $\sigma_1 = s(y_1) = [(\gamma_1^{-1}\alpha)\gamma_1]_{y_1} = [\beta_1]_{y_1}$, $\sigma_2 = s(y_2) = [(\gamma_2^{-1}\alpha)\gamma_2]_{y_2} = [\beta_2]_{y_2}$. If $\varphi^*(\sigma_1) = \varphi^*(\sigma_2)$, then $y_1 = y_2$, so $\gamma_1 \sim \gamma_2$ and $\sigma_1 = \sigma_2$.

2. $\varphi^* = \varphi \mid U$ is continuous. In fact, $\sigma \in U = s(W) \Rightarrow \varphi^*(\sigma) = y \in W$ and if $V = V(y) \subset W$ is a neighborhood, then $s(V) \subset U$ is a neighborhood of σ and $\varphi^*(s(V)) = V \subset W$. Hence $\varphi^*$is continuous.

3. $\varphi^{*-1} = (\varphi \mid U)^{-1} = s: W \to U$ is continuous. Indeed, if $y \in W$ is any point, $s(y) = \sigma \in U$ and $U' = U'(\sigma)$ is a neighborhood of σ, then $\varphi(U') \subset W$ is a neighborhood of y in W and $s(\varphi(U')) = U'$.

Therefore we can state the following theorem.

Theorem 1.1. Let X be a locally arcwise topological space and $\pi_1(X, x)$ be the corresponding fundamental group for each $x \in X$, and $H =$

$\bigvee_{x \in X} \pi_1(X, x)$. If the mapping $\varphi: H \to X$ is defined as above, then there exists a natural topology on H such that φ is locally topological with respect to this topology [1].

Thus the pair (H,φ) is a sheaf over X.

Definition 1.2. The sheaf (H,φ) given by the theorem 1.1. is called the sheaf of the fundamental groups of X or 1- dimensional homotopy groups of X

Definition 1.3. The fundamental group $\pi_1(X, x) = \varphi^{-1}(x)$ is called the stalk of the sheaf (H, φ) over X and denoted by H_x for every $x \in X$.

Now, if $x \in X$ is an arbitrarily fixed point and $W = W(x)$ is its arcwise connected open neighborhood, then let $\Gamma(W,H) = \{s: s: W \to H\}$. The set $\Gamma(X,H)$ is a group with the pointwise operation of multiplication. In fact, if $s_1, s_2 \in \Gamma(W,H)$ are obtained by means of the elements $[\alpha_1], [\alpha_2] \in \pi_1(X, x)$, respectively, then $s_1.s_2$ is obtained by means of the element $[\alpha_1. \alpha_2] \in \pi_1(X, x)$. It follows from this definition that the operation of multiplication is well - defined and closed. Clearly, the operation of multiplication is associative and the mapping $I: W \to H$ is the identity element which is obtained by means of the identity element of $\pi_1(X, x)$. On the other hand, if $s \in \Gamma(X,H)$ is obtained by means of the element $[\alpha] \in \pi_1(X, x)$, then the mapping $s^{-1} \in \Gamma(W,H)$, which is obtained by means of the element $[\alpha]^{-1}$, is the inverse element for $s \in \Gamma(W, H)$. Hence $\Gamma(W, H)$ is a group. Now, if the set $A \subset X$ is any open set, then $A = \bigcup_{i \in I} W_i$, where W_i is an arcwise connected open neighborhood for each $i \in I$. Hence we can define a mapping $s: A \to H$ as follows:

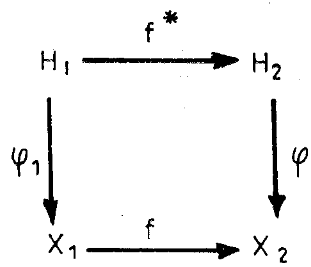
If $y \in A$ is an element, then there exists an arcwise connected open neighborhood W_i such that $y \in W_i$ and a mapping $s_i: W_i \to H$. Then, let $s(y) = s_i(y)$. Clearly, s is continuous and $\varphi \circ s = 1_A$. Hence the mapping s is called a section of H over A. Let us denote by $\Gamma(A,H)$ all of the sections of H over A. It is easily shown that $\Gamma(A,H)$ is a group with the pointwise operation of multiplication. Thus, the operation $(.): H \oplus H \to H$ (that is, $(\sigma_1, \sigma_2) \to \sigma_1. \sigma_2$, for every $\sigma_1, \sigma_2 \in H$) is continuous. Hence (H, φ) is an algebraic sheaf. It should be noticed that the stalks of (H, φ) may not be commutative.

2. Characterizations.

Let X_1, X_2 be any locally arcwise connected topological spaces and H_1, H_2 be the corresponding sheaves, respectively. Let us denote these as the pairs (X_1,H_1) and (X_2,H_2).

Definition 2.1. Let the pairs (X_1,H_1) and (X_2,H_2) be given. It is said that there is a homomorphism between these pairs and it is written $F=(f, f^*)$: $(X_1, H_1) \to (X_2,H_2)$, if there exists a pair $F=(f, f^*)$ such that

1. f: $X_1 \to X_2$ is a continuous mapping,
2. f^*: $H_1 \to H_2$ is a continuous mapping,
3. f^*preserves the stalks with respect to f. That is, the following square diagram is commutative.



4. For every $x_1 \in X_1$ the restricted map $f^* | (H_1)_{x_1}$: $(H_1)^{x_1} \to (H_2)^{f(x_1)}$ is a homomorphism.

Definition 2.2. Let the pairs (X_1,H_1) and (X_2,H_2) be given such that the map $F = (f, f^*)$: $(X_1, H_1) \to (X_2, H_2)$ is a homomorphism. Then the map $F=(f, f^*)$ is called an isomorphism and it is written $(X_1,H_1) \overset{F}{\underset{\to}{\sim}} (X_2, H_2)$, if the maps f^* and f are topological. Then the pairs (X_1, H_1) and (X_2, H_2) are called isomorphic.

Theorem 2.1. Let the pairs (X_1, H_1) and (X_2,H_2) be given. If the map f:$X_1 \to X_2$ is given as a continuous map, then there exists a homomorphism between the pairs (X_1, H_1) and (X_2, H_2).

Proof. Let $x_1 \in X_1$ be an arbitrarily fixed point. Then $f(x_1) \in X_2$ and $\pi_1(X_1, x_1) = (H_1)_{x_1} \subset H_1$, $\pi_2(X_2, f(x_1)) = (H_2)_{f(x_1)} \subset H_2$ are the corresponding stalks. If α_1, β_1 are two arcs at x_1, then the arcs α_2, β_2 can be defined as $\alpha_2 = f o \alpha_1, \beta_2 = f o \beta_1$, respectively. If $\alpha_1 \sim \beta_1$, then $\alpha_2 \sim \beta_2$. Thus the correspondence $[\alpha]_{x_1} \to [f o \alpha]_{f(x_1)}$ is well-defined, and maps the homotopy classes of arcs at x_1 to the homotopy classes of arcs at $f(x_1)$, that is, to each element $[\alpha]_{x_1}$ there corresponds a unique element $[fo\alpha]_{f(x_1)}$.

Since the point $x_1 \in X_1$ is arbitrarily fixed, the above correspondence gives us a map $f^*: H_1 \to H_2$ such that $f^*([\alpha]) = [fo\alpha] \in H_2$, for every $[\alpha] \in H_1$.

1. $f^*$is continuous. Because if $U_2 \subset H_2$ is any open set, then it can be shown that $f^{*-1}(U_2) = U_1 \subset H_1$ is an open set. In fact, if $U_2 \subset H_2$ is open, then $U_2 = \bigcup_{i\in I} s_i^2(W_i)$ and $\varphi_2(U_2) = \bigcup_{i\in I} W_i$, where the W_i's are arcwise open neighborhoods and the s_i^2's are sections over W_i. Thus, $\bigcup_{i\in I} W_i \subset X_2$ is open and $f^{-1}(\bigcup_{i\in I} W_i) = \bigcup_{i\in I} f^{-1}(W_i) \subset X_1$ is open, since f is continuous. Moreover, since $f^{-1}(W_i)$ is an arcwise connected open neighborhood in X_1 there exists a section $s_i^1: f^{-1}(W_i) \to H_1$. Hence $\bigcup_{i\in I} s_i^1(f^{-1}(W_i)) \subset H_1$ is an open set. Let us now show that $U_1 = \bigcup_{i\in I} s_i^1(f^{-1}(W_i))$. If $\sigma_1 = [\alpha]_{x_1} \in U_1 = f^{*-1}(U_2)$, then there exists a $\sigma_2 = [\beta]_{x_2} \in U_2 \ni f^*(\sigma_1) = \sigma_2$ and $\varphi_2(\sigma_2) = \varphi_2([\beta]_{x_2}) = x_2$, where $x_2 = f(x_1)$ and $\beta = fo\alpha$. Thus, if $x_2 \in W_i$, then $x_1 \in f^{-1}(W_i)$ and $\sigma_1 = [\alpha]_{x_1} \in \bigcup_{i\in I} s_i^1(W_i)$. Hence $U_1 \subset \bigcup_{i\in I} s_i^1(f^{-1}(W_i))$. On the other hand, $\sigma_1 \in \bigcup_{i\in I} s_i^1(f^{-1}(W_i))$ implies that $\sigma_1 \in s_i^1 f^{-1}(W_i)$ for some $i \in I$. From here, if $\sigma_1 = [\alpha]_{x_1}$, then $\varphi_1(\sigma_1) = x_1$, and $fo\alpha$ is a closed arc at $f(x_1) \in W_i$. Thus $[fo\alpha]_{f(x_1)} = \sigma_2 \in U_2$ and $\bigcup_{i\in I} s_i^1(f^{-1}(W_i)) \subset U_1$. Therefore, $U_1 = \bigcup_{i\in I} s_i^1(f^{-1}(W_i))$. Hence $f^*$is a continuous map.

2. f^*preserves the stalks with respect to f. In fact, for any $\sigma_1 = [\alpha]_{x_1} \in H_1$

$(fo\varphi_1)([\alpha]_{x_1}) = f(\varphi_1([\alpha]_{x_1})) = f(x_1)$.

$(\varphi_2 of)([\alpha]_{x_1}) = \varphi_2(f([\alpha]_{x_1})) = \varphi_2([fo\alpha]_{f(x_1)}) = f(x_1)$.

3. For every $x_1 \in X$ the map $f^* \mid (H_1)_{x_1}: (H_1)_{x_1} \to (H_2)_{f(x_1)}$ is homomorphism. In fact, if α_1, β_1 are two arcs at $x_1 \in X_1$, and $\alpha_2 = fo\alpha_1$, $\beta_2 = fo\beta_1$ are the carresponding arcs at $f(x_1) \in X_2$, then

$$\rho = \alpha_1 . \beta_1 = \begin{cases} \alpha(2t) & , 0 \leq t \leq 1/2 \\ \beta(2t-1) & , 1/2 \leq t \leq 1 \end{cases}$$

and

$$fo\rho = \begin{cases} f(\alpha(2t)) & , 0 \leq t \leq 1/2 \\ f(\beta(2t-1)) & , 1/2 \leq t \leq 1. \end{cases}$$

That is, $fo\ \rho = \alpha_2 . \beta_2$. Hence,

$f^*([\alpha_1]) . f^*([\beta_1]) = [\alpha_2]\ [\beta_2] = [\alpha_2\beta_2] = [fo\ \alpha_1 . \beta_1] =$
$f^*([\alpha_1\beta_1])$.

Therefore, if we write $F = (f, f^*)$, F is a homomorphism.

Now, we can give the following theorem.

Theorem 2.2. Let the pairs (X_1, H_1), (X_2, H_2), (X_3, H_3) and the continuous maps $f_1: X_1 \to X_2$, $f_2: X_2 \to X_3$ be given. Then, there exists a homomorphism $F = (f, f^*): (X_1, H_1) \to (X_3, H_3)$ such that $f = f_2\ of_1$, $f^* = f^*_2\ of^*_1$.

Proof. Since $f_2\ of_1: X_1 \to X_3$ is a continuous map, there exists a homomorphism $F = (f = f_2\ of_1, f^*): (X_1, H_1) \to (X_3, H_3)$ (Theorem. 3.1). To prove this theorem it is sufficient to show that $f^* = f^*_2 \text{ o } f^*_1$. However, for any $[\alpha] \in H_1$

$f^*([\alpha]) = [(f_2\ of_1) \text{ o } \alpha]$ and $(f^*_2\ of^*_1)([\alpha]) = f^*_2(f^*_1([\alpha])) = f^*_2([f_1 \text{ o } \alpha]) = [f_2 \text{ o } (f_1 \text{ o}\alpha)]$. Thus, we must show that $(f_2\ of_1) \text{ o } \alpha \sim f_1 \text{ o } (f_2 \text{ o } \alpha)$ rel. (0,1). Now, if α is a closed arc at $x_1 \in X_1$, then the maps $(f_2\ of_1)\ o\alpha: I \to X_3$ and $f_2 \text{ o } (f_1\ o\alpha): I \to X_3$ are continuous maps at $f_2(f_1(x)) \in X_3$. Let us define a map $F(t, k): IxJ \to X_3$ as follows:

$$F = F(t, k) = \begin{cases} (f_2 of_1)(\alpha(t)), & 0 \leq t \leq 1-k \\ f_2 o((f_1 o\ \alpha)(t)), & 1-k \leq t \leq 1. \end{cases}$$

It is clear that F is continuous. On the other hand $F(t,0) = (f_2 of_1) \text{ o } \alpha$, $F(t,1) = f_2 o(f_1 o\alpha)$ and $F(0, k) = F(1,k) = f_2(f_1(x_1))$. Hence, $(f_2 of_1)\ o\alpha \sim f_2 o\ (f_1 o\alpha)$ rel. (0.1).

Now, we can state the following theorem.

Theorem 2.3. There is a covariant functor from the category of the arcwise connected topological spaces and continuous maps to the category of sheaves and continuous homorphisms.

Theorem 2.4. Let the pairs (X_1, H_1) and (X_2, H_2) be given If the map $f: X_1 \to X_2$ is a topological map, then there exists an isomophism between the pairs (X_1, H_1) and (X_2, H_2).

Proof. By theoerm 3.1, there exists a homomorphism $F = (f, f^*)$ between the pairs (X_1, H_1) and (X_2, H_2). To prove this theorem it is sufficient to show that $f^*$is one -to- one and f^{*-1} is continuous.

Since f is a topological map it is continuous, one -to- one, and f^{-1} is continuous. So, Theorem 3.1. induces the homomorphisms $F = (f, f^*)$ and $F^{-1} = f^{-1}, (f^{-1})^*)$. On the other hand, for any two elements $[\alpha_1], [\alpha_2] \in H_1, f^*([\alpha_1]) = f^*([\alpha_2])$ implies that $[\beta_1] = [\beta_2] \Rightarrow (f^{-1})^*([\beta_1]) = (f^{-1})^*([\beta_2])$. Thus, $(f^{-1})^* (f^*([\alpha_1])) = (f^{-1})^* (f^*([\alpha_2]))$. Since $(f^{-1})^* o(f)^* = (f^{-1} of)^*$ and $f^{-1} of = 1_{X_1}, (f^{-1} of)^* = 1_{H_1}$ and $[\alpha_1] = [\alpha_2]$. Hence $f^*$is one -to- one. Since $f^{*-1} = (f^{-1})^*, f^{*-1}$ is continuous

Therefore, $F = (f, f^*)$ is an isomorphism.

ÖZET

Bu makalede, bir topolojik uzayın esas grubu vasıtası ile, bir cebirsel yapılı demet inşa edilmiş ve bazı cebirsel topolojik karakterizasyonlar verilmiştir.

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