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Research Article

Horizontal lifts of projectable linear connection to semi-tangent bundle

Furkan Yıldırım

Department of Mathematics, Faculty of Science, Atatürk University, 25240, Erzurum, Turkey

Abstract

The main aim of this article is to study the horizontal lifts of projectable linear connection in the semi-tangent bundle tM. The properties of complete and horizontal lifts of projectable linear connection for semi-tangent bundle tM are also investigated. Finally, we examine the infinitesimal linear transformation in the semi-tangent bundle with respect to the horizontal lift of a projectable linear connection.

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1. Pullback bundle of the tangent bundle

Let M_n be an $n-$dimensional differentiable manifold of class C^∞ and finite dimension n, and let (M_n, π_1, B_m) be a differentiable bundle over B_m. We use the notation $(x^i) = (x^a, x^\alpha)$, where the indices $i, j, ...$ have range in $\{1, 2, ..., n\}$, the indices $a, b, ...$ have range in $\{1, 2, ..., n-m\}$ and the indices $\alpha, \beta, ...$have range in $\{n-m+1, n-m+2, ..., n\}$, x^α are coordinates in B_m , x^a are fiber coordinates of the bundle

$$\pi_1 : M_n \to B_m.$$

Let now $(T(B_m), \widetilde{\pi}, B_m)$ be a tangent bundle [14] over base space B_m, and let M_n be differentiable bundle determined by a submersion (natural projection) $\pi_1 : M_n \to B_m$. The semi-tangent bundle (pullback [3,5,9,10,15,16]) of the tangent bundle $(T(B_m), \widetilde{\pi}, B_m)$ is the bundle $(t(B_m), \pi_2, M_n)$ over differentiable bundle M_n with a total space

$$t(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times T_x(B_m) : \pi_1 (x^a, x^\alpha) = \widetilde{\pi} \left(x^\alpha, x^{\overline{\alpha}} \right) = (x^\alpha) \right\} \subset M_n \times T_x(B_m)$$

and with the projection map $\pi_2 : t(B_m) \to M_n$ defined by $\pi_2(x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha)$, where $T_x(B_m)$ is the tangent space at a point x of B_m $(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$, where $x^{\overline{\alpha}} = y^\alpha \left(\overline{\alpha}, \overline{\beta}, ... = n+1, ..., 2n \right)$ are fiber coordinates of the tangent bundle $T(B_m)$.

Where the pullback (or Pontryagin [7]) bundle $t(B_m)$ of the differentiable bundle M_n also has the natural bundle structure over B_m, its bundle projection $\pi : t(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$, and hence $\pi = \pi_1 \circ \pi_2$. Thus $(t(B_m),\pi_1 \circ \pi_2)$ is

Email address: furkan.yildirim@atauni.edu.tr
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the step-like bundle [6] or composite bundle [8, p. 9]. Consequently, we notice the semi-tangent bundle $(t(B_m),\pi_2\ )$ is a pullback bundle of the tangent bundle over B_m by π_1 [9].

If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another local adapted coordinates in differentiable bundle M_n, then we get:

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right) \ . \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has the components

$$\left(A_j^{i'}\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_b^{a'} = \frac{\partial x^{a'}}{\partial x^b}$, $A_\beta^{a'} = \frac{\partial x^{a'}}{\partial x^\beta}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$ [9].

To a transformation (1.1) of local coordinates of M_n, there corresponds on $t(B_m)$ the change of coordinate

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta. \end{cases} \tag{1.2}$$

The Jacobian of (1.2) is:

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix}, \tag{1.3}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J, ... = 1, ..., 2n$; $A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}$ [9].

The main aim of this article is to study the horizontal lifts of projectable linear connection to semi-tangent (pullback) bundle $(t(B_m)\ , \pi_2)$ and their properties.

We denote by $\Im_q^p(M_n)$ the module over $F\left(M_n\right)$ of all tensor fields of type (p, q) on M_n, where $F\left(M_n\right)$ is the algebra of $C^\infty -$ functions on M_n.

2. Some lifts of tensor fields of types (1,0) and (0,1)

If f is a function on B_m, we write ${}^{vv}f$ for the function on $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \to B_m$ and ${}^v f = f \circ \pi_1$, so that

$${}^{vv}f = {}^v f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi.$$

Thus, the vertical lift ${}^{vv}f$ of the function f to $t(B_m)$ satisfies

$${}^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha). \tag{2.1}$$

We note here that value ${}^{vv}f$ is constant along each fibre of $\pi : t(B_m) \to B_m$.

Let $X \in \Im_0^1(B_m)$, i.e. $X = X^\alpha \partial_\alpha$. On putting

$${}^{vv}X = ({}^{vv}X^\alpha) = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{2.2}$$

from (1.3), one can easily prove that ${}^{vv}X' = \bar{A}({}^{vv}X)$. The vector field ${}^{vv}X$ is called the vertical lift of X to $t(B_m)$.

Let $\omega \in \Im_1^0(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting

$${}^{vv}\omega = \left({}^{vv}\omega\right)_\alpha = \left(0, \omega_\alpha, 0\right), \tag{2.3}$$

from (1.3), we easily see that ${}^{vv}\omega = \bar{A}\,{}^{vv}\omega'$. The covector field ${}^{vv}\omega$ is called the vertical lift of ω to $t(B_m)$.

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [12] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. Now, consider $\widetilde{X} \in \Im_0^1(M_n)$, the complete lift ${}^{cc}\widetilde{X}$ of $\widetilde{X}$ to the semi-tangent bundle $t(B_m)$ has components [9]:

$$ {}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^\alpha\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{2.4} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

For any $F \in \Im_1^1(B_m)$, from (1.3), we have $(\gamma F)' = \bar{A}(\gamma F)$, where γF is a vector field in $t(B_m)$ defined by

$$ \gamma F = (\gamma F^I) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{2.5} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

Let now $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field on M_n with projection $X \in \Im_0^1(B_m)$ [12]. The horizontal lift ${}^{HH}\widetilde{X}$ of $\widetilde{X}$ on $t(M_n)$ is defined by:

$$ {}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X}). $$

Where ∇ is a projectable symmetric linear connection in a differentiable manifold B_m. Then, remembering that ${}^{cc}\widetilde{X}$ and $\gamma(\nabla\widetilde{X})$ have, respectively, local components

$$ {}^{cc}\widetilde{X} = \left({}^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, \gamma(\nabla\widetilde{X}) = \left(\gamma(\nabla\widetilde{X})^I\right) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$ (\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma_{\beta\alpha}^\varepsilon. $$

We find that the horizontal lift ${}^{HH}\widetilde{X}$ of $\widetilde{X}$ has the components

$$ {}^{HH}X = \left({}^{HH}X^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma_\beta^\alpha X^\beta \end{pmatrix} \tag{2.6} $$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. Where

$$ \Gamma_\beta^\alpha = y^\varepsilon \Gamma_{\varepsilon\,\beta}^\alpha. \tag{2.7} $$

3. Complete lifts of projectable linear connection

Let $\Gamma_{\alpha\gamma}^\beta$ be components of projectable linear connection [1, 2, 12, 13] ∇ with respect to local coordinates (x^α) in B_m and ${}^{cc}\Gamma_{I\,K}^J$ components of ${}^{cc}\nabla$ with respect to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $t(B_m)$.

We recall from [12] that components ${}^{cc}\Gamma_{I\,K}^J$ of complete lift ${}^{cc}\nabla$ of projectable linear connection ∇ can be calculated from base manifold B_m to semi-tangent bundle $t(B_m)$ also as:

$$\begin{cases} {}^{cc}\Gamma^{b}_{ac} = {}^{cc}\Gamma^{b}_{a\gamma} = {}^{cc}\Gamma^{b}_{a\overline{\gamma}} = {}^{cc}\Gamma^{b}_{\alpha c} = {}^{cc}\Gamma^{b}_{\alpha\overline{\gamma}} = {}^{cc}\Gamma^{b}_{\overline{\alpha} c} = {}^{cc}\Gamma^{b}_{\overline{\alpha}\gamma} = {}^{cc}\Gamma^{b}_{\overline{\alpha}\overline{\gamma}} = 0, \\ {}^{cc}\Gamma^{b}_{\alpha\gamma} = \Gamma^{b}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\beta}_{ac} = {}^{cc}\Gamma^{\beta}_{a\gamma} = {}^{cc}\Gamma^{\beta}_{a\overline{\gamma}} = {}^{cc}\Gamma^{\beta}_{\alpha c} = {}^{cc}\Gamma^{\beta}_{\alpha\overline{\gamma}} = {}^{cc}\Gamma^{\beta}_{\overline{\alpha} c} = {}^{cc}\Gamma^{\beta}_{\overline{\alpha}\gamma} = {}^{cc}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}} = 0, \\ {}^{cc}\Gamma^{\beta}_{\alpha\gamma} = \Gamma^{\beta}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{ac} = {}^{cc}\Gamma^{\overline{\beta}}_{a\gamma} = {}^{cc}\Gamma^{\overline{\beta}}_{a\overline{\gamma}} = {}^{cc}\Gamma^{\overline{\beta}}_{\alpha c} = {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha} c} = {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}} = 0, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}} = \Gamma^{\beta}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\alpha\gamma} = y^{\varepsilon}\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma} = \Gamma^{\beta}_{\alpha\gamma}. \end{cases} \tag{3.1}$$

Where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $K = (c, \gamma, \overline{\gamma})$.

On the other hand, from (3.1) we obtain:

Theorem 3.1. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X \in \Im^1_0(B_m)$ and $Y \in \Im^1_0(B_m)$, respectively. We have:*

(i) ${}^{cc}\nabla_{{}^{vv}X}({}^{vv}Y) = 0,$
(ii) ${}^{cc}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y}) = 0,$
(iii) ${}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y) = {}^{vv}\left(\nabla_X Y\right),$
(iv) ${}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}\left(\nabla_X Y\right) + \gamma(R(\ ,X)Y),$
(v) $[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}] = {}^{cc}[\widetilde{X}, \widetilde{Y}] (i.e. L_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y}) = {}^{cc}\left(L_{\widetilde{X}}\widetilde{Y}\right)),$
(vi) $[{}^{cc}\widetilde{X}, \gamma F] = \gamma\left(L_X F\right)(F \in \Im^1_1(B_m)),$

where $R(\ ,X)Y \in \Im^1_1(B_m)$ is a tensor field of type of $(1,1)$ defined by $F(Z) = R(Z,X)Y$ for any $Z \in \Im^1_0(B_m)$ and L_X is the operator of Lie derivation with respect to X.

4. Horizontal lifts of projectable linear connection

Let there be given a projectable linear connection ∇ in B_m. We shall now define the horizontal lift ${}^{HH}\nabla$ of a projectable linear connection ∇ in B_m to $t(B_m)$ by the conditions:

(i) ${}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y) = 0,$
(ii) ${}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y}) = 0,$
(iii) ${}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y) = {}^{vv}\left(\nabla_X Y\right),$
(iv) ${}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}\left(\nabla_X Y\right),$

for any $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$.

Thus, if we put

$$\widetilde{S}(\widetilde{X}, \widetilde{Y}) = {}^{HH}\nabla_{\widetilde{X}}\widetilde{Y} - {}^{cc}\nabla_{\widetilde{X}}\widetilde{Y} \tag{4.1}$$

for any $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$. Then, from (4.1) and Theorem 3.1, the tensor $\widetilde{S}$ of type $(1,2)$ in $t(B_m)$ satisfies the conditions

(i) $\widetilde{S}({}^{vv}X, {}^{vv}Y) = 0,$
(ii) $\widetilde{S}({}^{vv}X, {}^{HH}\widetilde{Y}) = 0,$
(iii) $\widetilde{S}({}^{HH}\widetilde{X}, {}^{vv}Y) = 0$
(iv) $\widetilde{S}({}^{HH}\widetilde{X}, {}^{HH}\widetilde{Y}) = -\gamma(R(\ ,X)Y),$

for any $\widetilde{X}, \widetilde{Y} \in \Im^1_0(M_n)$. Therefore $\widetilde{S}$ has the components $\widetilde{S}^J_{IK}$ such that

$$\widetilde{S}^{\overline{\beta}}_{\alpha\gamma} = -y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma} \tag{4.2}$$

all others being zero, with respect to the induced coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t(B_m)$.

Since the components ${}^{cc}\Gamma^J_{I\,K}$ of ${}^{cc}\nabla$ are given by (3.1), it follows from (4.1) and (4.2) that the horizontal lift ${}^{HH}\nabla$ of a projectable linear connection ∇ has the components ${}^{HH}\Gamma^J_{I\,K}$ such that

$$\begin{cases} {}^{HH}\Gamma^b_{ac} = {}^{HH}\Gamma^b_{a\gamma} = {}^{HH}\Gamma^b_{a\overline{\gamma}} = {}^{HH}\Gamma^b_{\alpha c} = {}^{HH}\Gamma^b_{\alpha\overline{\gamma}} = {}^{HH}\Gamma^b_{\overline{\alpha}c} = {}^{HH}\Gamma^b_{\overline{\alpha}\gamma} = {}^{HH}\Gamma^b_{\overline{\alpha}\overline{\gamma}} = 0, \\ {}^{HH}\Gamma^b_{\alpha\gamma} = \Gamma^b_{\alpha\gamma}, \\ {}^{HH}\Gamma^\beta_{ac} = {}^{HH}\Gamma^\beta_{a\gamma} = {}^{HH}\Gamma^\beta_{a\overline{\gamma}} = {}^{HH}\Gamma^\beta_{\alpha c} = {}^{HH}\Gamma^\beta_{\alpha\overline{\gamma}} = {}^{HH}\Gamma^\beta_{\overline{\alpha}c} = {}^{HH}\Gamma^\beta_{\overline{\alpha}\gamma} = {}^{HH}\Gamma^\beta_{\overline{\alpha}\overline{\gamma}} = 0, \\ {}^{HH}\Gamma^\beta_{\alpha\gamma} = \Gamma^\beta_{\alpha\gamma}, \\ {}^{HH}\Gamma^{\overline{\beta}}_{ac} = {}^{HH}\Gamma^{\overline{\beta}}_{a\gamma} = {}^{HH}\Gamma^{\overline{\beta}}_{a\overline{\gamma}} = {}^{HH}\Gamma^{\overline{\beta}}_{\alpha c} = {}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}c} = {}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}} = 0, \\ {}^{HH}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}} = \Gamma^\beta_{\alpha\gamma}, \\ {}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma} = y^\varepsilon\partial_\varepsilon\Gamma^\beta_{\alpha\gamma} - y^\varepsilon R^\beta_{\varepsilon\alpha\gamma}, \\ {}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma} = \Gamma^\beta_{\alpha\gamma}. \end{cases} \tag{4.3}$$

with respect to the induced coordinates in $t(B_m)$. Where ${}^{HH}\Gamma^J_{I\,K}$ are local components of ${}^{HH}\nabla$ in $t(B_m)$.

Proof. For convenience sake, we only consider ${}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma}$. According to (3.1), (4.1) and (4.2), in fact:

$$\begin{aligned} \widetilde{S}^{\overline{\beta}}_{\alpha\gamma} &= {}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma} - {}^{cc}\Gamma^{\overline{\beta}}_{\alpha\gamma} \\ -y^\varepsilon R_{\varepsilon\alpha\gamma} &= {}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma} - y^\varepsilon\partial_\varepsilon\Gamma^\beta_{\alpha\gamma} \\ {}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma} &= y^\varepsilon\partial_\varepsilon\Gamma^\beta_{\alpha\gamma} - y^\varepsilon R^\beta_{\varepsilon\alpha\gamma}. \end{aligned}$$

Thus, we have ${}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma} = y^\varepsilon\partial_\varepsilon\Gamma^\beta_{\alpha\gamma} - y^\varepsilon R^\beta_{\varepsilon\alpha\gamma}$. In a similar way, we can easily find other components of ${}^{HH}\Gamma^J_{I\,K}$. □

Theorem 4.1. *Let $\widetilde{X}$ be a projectable vector field on M_n with projections X on B_m. If $Y \in \Im^1_0(B_m)$, then*

$${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y) = {}^{vv}\left(\nabla_X Y\right).$$

Proof. If $\widetilde{X} \in \Im^1_0(M_n)$, $Y \in \Im^1_0(B_m)$ and $\begin{pmatrix} \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^b \\ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^\beta \\ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^{\overline{\beta}} \end{pmatrix}$ are the components of $\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(B_m)$, then we find

$$\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^J = {}^{cc}\widetilde{X}^a\,{}^{HH}\nabla_a({}^{vv}Y)^J + {}^{cc}\widetilde{X}^\alpha\,{}^{HH}\nabla_\alpha({}^{vv}Y)^J + {}^{cc}\widetilde{X}^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^J.$$

As the first coordinate, if $J = b$, we obtain

$$\begin{aligned} \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^b &= {}^{cc}\widetilde{X}^a\,{}^{HH}\nabla_a({}^{vv}Y)^b + {}^{cc}\widetilde{X}^\alpha\,{}^{HH}\nabla_\alpha({}^{vv}Y)^b + {}^{cc}\widetilde{X}^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^b \\ &= X^a(\partial_a \underbrace{{}^{vv}Y^b}_{0} + {}^{HH}\Gamma^b_{ac}\underbrace{{}^{vv}Y^c}_{0} + {}^{HH}\Gamma^b_{a\gamma}\underbrace{{}^{vv}Y^\gamma}_{0} + \underbrace{{}^{HH}\Gamma^b_{a\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}}) \\ &\quad + X^\alpha(\partial_\alpha \underbrace{{}^{vv}Y^b}_{0} + {}^{HH}\Gamma^b_{\alpha c}\underbrace{{}^{vv}Y^c}_{0} + {}^{HH}\Gamma^b_{\alpha\gamma}\underbrace{{}^{vv}Y^\gamma}_{0} + \underbrace{{}^{HH}\Gamma^b_{\alpha\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}}) \\ &\quad + {}^{cc}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}} \underbrace{{}^{vv}Y^b}_{0} + {}^{HH}\Gamma^b_{\overline{\alpha}c}\underbrace{{}^{vv}Y^c}_{0} + {}^{HH}\Gamma^b_{\overline{\alpha}\gamma}\underbrace{{}^{vv}Y^\gamma}_{0} + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}}) \\ &= 0 \end{aligned}$$

by virtue of (2.2), (2.4) and (4.3). As the second coordinate, if $J=\beta$, we obtain

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^{\beta} &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{vv}Y)^{\beta}+{}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{vv}Y)^{\beta}+{}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^{\beta}\\ &= X^a(\partial_a\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{HH}\Gamma^{\beta}_{ac}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma^{\beta}_{a\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma^{\beta}_{a\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\ &\quad+X^\alpha(\partial_\alpha\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{HH}\Gamma^{\beta}_{\alpha c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma^{\beta}_{\alpha\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\ &\quad+{}^{cc}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{HH}\Gamma^{\beta}_{\overline{\alpha}c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\ &= 0\end{aligned}$$

by virtue of (2.2), (2.4) and (4.3). As the third coordinate, if $J=\overline{\beta}$, then we obtain

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)\right)^{\overline{\beta}} &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{vv}Y)^{\overline{\beta}}+{}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{vv}Y)^{\overline{\beta}}+{}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^{\overline{\beta}}\\ &= X^a(\underbrace{\partial_a({}^{vv}Y)^{\overline{\beta}}}_{0}+{}^{HH}\Gamma^{\overline{\beta}}_{ac}({}^{vv}Y)^{c}+{}^{HH}\Gamma^{\overline{\beta}}_{a\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{a\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\ &\quad+X^\alpha(\partial_\alpha\underbrace{{}^{vv}Y^{\overline{\beta}}}_{Y^{\beta}}+{}^{HH}\Gamma^{\overline{\beta}}_{\alpha c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}}}_{\Gamma^{\beta}_{\alpha\gamma}}({}^{vv}Y)^{\overline{\gamma}})\\ &\quad+{}^{cc}\widetilde{X}^{\overline{\alpha}}(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\ &= X^\alpha\partial_\alpha Y^\beta+X^\alpha\Gamma^{\beta}_{\alpha\gamma}Y^\gamma=X^\alpha\left(\partial_\alpha Y^\beta+\Gamma^{\beta}_{\alpha\gamma}Y^\gamma\right)\\ &= (\nabla_X Y)^\beta\end{aligned}$$

by virtue of (2.2), (2.4) and (4.3). Therewithal, we know that ${}^{vv}(\nabla_X Y)$ have the components

$${}^{vv}(\nabla_X Y)=\begin{pmatrix}0\\0\\(\nabla_X Y)^\beta\end{pmatrix}$$

with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t(B_m)$. Thus, we have ${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}Y)={}^{vv}(\nabla_X Y)$ in $t(B_m)$. □

Theorem 4.2. *Let $\widetilde{X}$ be a projectable vector field on M_n with projections X on B_m. If $\omega\in\Im^0_1(B_m)$, then*

$${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)={}^{vv}(\nabla_X\omega).$$

Proof. If $\widetilde{X}\in\Im^1_0(M_n)$, $\omega\in\Im^0_1(B_m)$ and $\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)_c,{}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)_\gamma,{}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)_{\overline{\gamma}}\right)$ are the components of $\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)\right)_K$ with respect to the coordinates $(x^c,x^\gamma,x^{\overline{\gamma}})$ on $t(B_m)$, then we find

$$\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)\right)_K={}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{vv}\omega)_K+{}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{vv}\omega)_K+{}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_K.$$

As the first coordinate, if $K=c$, we obtain

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)\right)_K &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{vv}\omega)_K+{}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{vv}\omega)_K+{}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_K\\ &= X^a(\partial_a\underbrace{{}^{vv}\omega_c}_{0}-{}^{HH}\Gamma^{b}_{ac}\underbrace{({}^{vv}\omega)_b}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{ac}}_{0}({}^{vv}\omega)_\beta-{}^{HH}\Gamma^{\overline{\beta}}_{ac}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0})\end{aligned}$$

$$
\begin{aligned}
& +X^{\alpha}(\partial_{\alpha}\underbrace{{}^{vv}\omega_{c}}_{0}-{}^{HH}\Gamma^{b}_{\alpha c}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha c}}_{0}({}^{vv}\omega)_{\beta}-{}^{HH}\,\Gamma^{\overline{\beta}}_{\alpha c}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
& +{}^{cc}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}\omega_{c}}_{0}-{}^{HH}\Gamma^{b}_{\overline{\alpha}c}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}c}}_{0}({}^{vv}\omega)_{\beta}-{}^{HH}\,\Gamma^{\overline{\beta}}_{\overline{\alpha}c}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
= {} & 0
\end{aligned}
$$

by virtue of (2.3), (2.4) and (4.3). As the second coordinate, if $K=\gamma$, we obtain

$$
\begin{aligned}
\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)\right)_{\gamma} &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_{a}({}^{vv}\omega)_{\gamma}+{}^{cc}\,\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{vv}\omega)_{\gamma}+{}^{cc}\,\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_{\gamma} \\
&= {}^{cc}\widetilde{X}^{a}(\partial_{a}\underbrace{{}^{vv}\omega_{\gamma}}_{0}-{}^{HH}\Gamma^{b}_{a\gamma}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{a\gamma}}_{0}({}^{vv}\omega)_{b}-{}^{HH}\,\Gamma^{\overline{\beta}}_{a\gamma}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
&\quad +{}^{cc}\widetilde{X}^{\alpha}(\partial_{\alpha}\underbrace{{}^{vv}\omega_{\gamma}}_{0}-{}^{HH}\Gamma^{b}_{\alpha\gamma}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\gamma}}_{\Gamma^{\beta}_{\alpha\gamma}}\underbrace{{}^{vv}\omega_{b}}_{\omega_{b}}-{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
&\quad +{}^{cc}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}\omega_{\gamma}}_{0}-{}^{HH}\Gamma^{b}_{\overline{\alpha}\gamma}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\gamma}}_{0}({}^{vv}\omega)_{b}-{}^{HH}\,\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
&= X^{\alpha}\partial_{\alpha}\omega_{\gamma}-X^{\alpha}\Gamma^{\beta}_{\alpha\gamma}\omega_{\beta} \\
&= X^{\alpha}\left(\partial_{\alpha}\omega_{\gamma}-\Gamma^{\beta}_{\alpha\gamma}\omega_{\beta}\right) \\
&= (\nabla_{X}\omega)_{\gamma}
\end{aligned}
$$

by virtue of (2.3), (2.4) and (4.3). As the third coordinate, if $K=\overline{\gamma}$, then we obtain

$$
\begin{aligned}
\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)\right)_{\overline{\gamma}} &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_{a}({}^{vv}\omega)_{\overline{\gamma}}+{}^{cc}\,\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{vv}\omega)_{\overline{\gamma}}+{}^{cc}\,\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_{\overline{\gamma}} \\
&= X^{a}(\partial_{a}\underbrace{{}^{vv}\omega_{\overline{\gamma}}}_{0}-{}^{HH}\Gamma^{b}_{a\overline{\gamma}}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{a\overline{\gamma}}}_{0}({}^{vv}\omega)_{\beta}-{}^{HH}\,\Gamma^{\overline{\beta}}_{a\overline{\gamma}}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
&\quad +X^{\alpha}(\partial_{\alpha}\underbrace{{}^{vv}\omega_{\overline{\gamma}}}_{0}-{}^{HH}\Gamma^{b}_{\alpha\overline{\gamma}}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\overline{\gamma}}}_{0}({}^{vv}\omega)_{\beta}-{}^{HH}\,\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
&\quad +{}^{cc}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}\omega_{\overline{\gamma}}}_{0}-{}^{HH}\Gamma^{b}_{\overline{\alpha}\overline{\gamma}}\underbrace{{}^{vv}\omega_{b}}_{0}-\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}\omega)_{\beta}-{}^{HH}\,\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0}) \\
&= 0
\end{aligned}
$$

by virtue of (2.3), (2.4) and (4.3). Therewithal, we know that ${}^{vv}\,(\nabla_{X}\omega)$ have the components

$$
{}^{vv}\,(\nabla_{X}\omega)=(0,(\nabla_{X}\omega)_{\gamma}\,,0)
$$

with respect to the coordinates $(x^{c},x^{\gamma},x^{\overline{\gamma}})$ on $t(B_{m})$. Thus, we have ${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{vv}\omega)={}^{vv}\,(\nabla_{X}\omega)$ in $t(B_{m})$. □

Theorem 4.3. *Let $X\in\Im^{1}_{0}(B_{m})$. If $\omega\in\Im^{0}_{1}(B_{m})$, then*

$$
{}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)=0.
$$

Proof. If $X\in\Im^{1}_{0}(B_{m})$, $\omega\in\Im^{0}_{1}(B_{m})$ and $\left(\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_{c},\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_{\gamma},\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_{\overline{\gamma}}\right)$ are the components of $\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_{K}$ with respect to the coordinates $(x^{c},x^{\gamma},x^{\overline{\gamma}})$ on $t(B_{m})$, then we have

$$
\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_{K}={}^{vv}\,X^{a\,HH}\nabla_{a}({}^{vv}\omega)_{K}+{}^{vv}\,X^{\alpha\,HH}\nabla_{\alpha}({}^{vv}\omega)_{K}+{}^{vv}\,X^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_{K}.
$$

As the first coordinate, if $K=c$, we obtain

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_c &= \underbrace{{}^{vv}X^a}_{0}\,{}^{HH}\nabla_a({}^{vv}\omega)_c + \underbrace{{}^{vv}X^\alpha}_{0}\,{}^{HH}\nabla_\alpha({}^{vv}\omega)_c + {}^{vv}X^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_c\\ &= {}^{vv}X^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}\omega_c}_{0} - {}^{HH}\Gamma^b_{\overline{\alpha}c}\underbrace{{}^{vv}\omega_b}_{0} - \underbrace{{}^{HH}\Gamma^\beta_{\overline{\alpha}c}}_{0}({}^{vv}\omega)_\beta - {}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0})\\ &= 0\end{aligned}$$

by virtue of (2.2), (2.3) and (4.3). As the second coordinate, if $K=\gamma$, we obtain

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_\gamma &= \underbrace{{}^{vv}X^a}_{0}\,{}^{HH}\nabla_a({}^{vv}\omega)_\gamma + \underbrace{{}^{vv}X^\alpha}_{0}\,{}^{HH}\nabla_\alpha({}^{vv}\omega)_\gamma + \underbrace{{}^{vv}X^{\overline{\alpha}}}_{X^\alpha}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_\gamma\\ &= X^\alpha(\partial_{\overline{\alpha}}\underbrace{{}^{vv}\omega_\gamma}_{0} - {}^{HH}\Gamma^b_{\overline{\alpha}\gamma}\underbrace{{}^{vv}\omega_b}_{0} - \underbrace{{}^{HH}\Gamma^\beta_{\overline{\alpha}\gamma}}_{0}({}^{vv}\omega)_\beta - {}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0})\\ &= 0\end{aligned}$$

by virtue of (2.2), (2.3) and (4.3). As the third coordinate, if $K=\overline{\gamma}$, then we obtain

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)\right)_{\overline{\gamma}} &= \underbrace{{}^{vv}X^a}_{0}\,{}^{HH}\nabla_a({}^{vv}\omega)_{\overline{\gamma}} + \underbrace{{}^{vv}X^\alpha}_{0}\,{}^{HH}\nabla_\alpha({}^{vv}\omega)_{\overline{\gamma}} + {}^{vv}X^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{vv}\omega)_{\overline{\gamma}}\\ &= {}^{vv}X^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}\omega_{\overline{\gamma}}}_{0} - {}^{HH}\Gamma^b_{\overline{\alpha}\overline{\gamma}}\underbrace{{}^{vv}\omega_b}_{0} - \underbrace{{}^{HH}\Gamma^\beta_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{vv}\omega)_\beta - {}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}\underbrace{{}^{vv}\omega_{\overline{\beta}}}_{0})\\ &= 0\end{aligned}$$

by virtue of (2.2), (2.3) and (4.3). Thus, we have ${}^{HH}\nabla_{{}^{vv}X}({}^{vv}\omega)=0$. □

Theorem 4.4. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X\in\Im^1_0(B_m)$ and $Y\in\Im^1_0(B_m)$, respectively. We have:*

$$^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}(\nabla_X Y).$$

Proof. If $\widetilde{X},\widetilde{Y}\in\Im^1_0(M_n)$ and $\begin{pmatrix}\left({}^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^b\\ \left({}^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^\beta\\ \left({}^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}}\end{pmatrix}$ are the components of $\left({}^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^J$ with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t(B_m)$, then we find

$$\left({}^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^J = {}^{HH}\widetilde{X}^a\,{}^{HH}\nabla_a({}^{HH}\widetilde{Y})^J + {}^{HH}\widetilde{X}^\alpha\,{}^{HH}\nabla_\alpha({}^{HH}\widetilde{Y})^J + {}^{HH}\widetilde{X}^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^J.$$

As the first coordinate, if $J=b$, we obtain

$$\begin{aligned}&\left({}^{HH}\nabla_{^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^b\\ &= {}^{HH}\widetilde{X}^a\,{}^{HH}\nabla_a({}^{HH}\widetilde{Y})^b + {}^{HH}\widetilde{X}^\alpha\,{}^{HH}\nabla_\alpha({}^{HH}\widetilde{Y})^b + {}^{HH}\widetilde{X}^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^b\\ &= X^a\,{}^{HH}\nabla_a({}^{HH}\widetilde{Y})^b + X^\alpha\,{}^{HH}\nabla_\alpha({}^{HH}\widetilde{Y})^b + \left(-y^\varepsilon\Gamma^\alpha_{\varepsilon\phi}X^\phi\right){}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^b\\ &= X^a(\underbrace{\partial_a Y^b}_{0} + \underbrace{{}^{HH}\Gamma^b_{ac}}_{0}\,{}^{HH}Y^c + \underbrace{{}^{HH}\Gamma^b_{a\gamma}}_{0}\,{}^{HH}Y^\gamma + \underbrace{{}^{HH}\Gamma^b_{a\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y})^{\overline{\gamma}})\\ &\quad + X^\alpha(\partial_\alpha Y^b + \underbrace{{}^{HH}\Gamma^b_{\alpha c}}_{0}\,{}^{HH}Y^c + \underbrace{{}^{HH}\Gamma^b_{\alpha\gamma}}_{\Gamma^b_{\alpha\gamma}}\,{}^{HH}Y^\gamma + \underbrace{{}^{HH}\Gamma^b_{\alpha\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y})^{\overline{\gamma}})\\ &\quad + \left(-y^\varepsilon\Gamma^\alpha_{\varepsilon\phi}X^\phi\right)(\underbrace{\partial_{\overline{\alpha}}Y^b}_{0} + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}c}}_{0}\,{}^{HH}Y^c + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}\gamma}}_{0}\,{}^{HH}Y^\gamma + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y})^{\overline{\gamma}})\end{aligned}$$

$$
\begin{aligned}
&= X^{\alpha}\partial_{\alpha}Y^{b} + X^{\alpha}\Gamma^{b}_{\alpha\gamma}Y^{\gamma} \\
&= X^{\alpha}\left(\partial_{\alpha}Y^{b} + \Gamma^{b}_{\alpha\gamma}Y^{\gamma}\right) \\
&= {}^{HH}\left(\nabla_{X}Y\right)^{b}
\end{aligned}
$$

by virtue of (2.6) and (4.3). As the second coordinate, if $J = \beta$, we obtain

$$
\begin{aligned}
&\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\beta} \\
&={}^{HH}\ \widetilde{X}^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\beta} + {}^{HH}\ \widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\beta} + {}^{HH}\ \widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\beta} \\
&= X^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\beta} + X^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\beta} + \left(-y^{\varepsilon}\Gamma^{\alpha}_{\varepsilon\,\phi}X^{\phi}\right){}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\beta} \\
&= X^{a}(\underbrace{\partial_{a}Y^{\beta}}_{0} + \underbrace{{}^{HH}\Gamma^{\beta}_{a\,c}}_{0}Y^{c} + \underbrace{{}^{HH}\Gamma^{\beta}_{a\,\gamma}}_{0}Y^{\gamma} + \underbrace{{}^{HH}\Gamma^{\beta}_{a\,\overline{\gamma}}}_{0}Y^{\overline{\gamma}}) \\
&\quad + X^{\alpha}(\partial_{\alpha}Y^{\beta} + \underbrace{{}^{HH}\Gamma^{\beta}_{\alpha c}}_{0}Y^{c} + \underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\gamma}}_{\Gamma^{\beta}_{\alpha\gamma}}Y^{\gamma} + \underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\overline{\gamma}}}_{0}Y^{\overline{\gamma}}) \\
&\quad + \left(-y^{\varepsilon}\Gamma^{\alpha}_{\varepsilon\,\phi}X^{\phi}\right)(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0} + \underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}c}}_{0}Y^{c} + \underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\gamma}}_{0}Y^{\gamma} + \underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}}}_{0}Y^{\overline{\gamma}}) \\
&= X^{\alpha}\partial_{\alpha}Y^{\beta} + X^{\alpha}\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma} \\
&= X^{\alpha}\left(\partial_{\alpha}Y^{\beta} + \Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}\right) \\
&= {}^{HH}\left(\nabla_{X}Y\right)^{\beta}
\end{aligned}
$$

by virtue of (2.6) and (4.3). As the third coordinate, if $J = \overline{\beta}$, then we obtain

$$
\begin{aligned}
&\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}} \\
&={}^{HH}\ \widetilde{X}^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\overline{\beta}} + {}^{HH}\ \widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\overline{\beta}} + {}^{HH}\ \widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\overline{\beta}} \\
&\quad + {}^{HH}\ \widetilde{X}^{\alpha}(\partial_{\alpha}\left(-y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\,\phi}Y^{\phi}\right) + \underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha c}}_{0}{}^{HH}Y^{c} \\
&\quad + \underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma}{}^{HH}Y^{\gamma}}_{y^{\varepsilon}\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma} - y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}Y^{\gamma}} + \underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}}}_{\Gamma^{\beta}_{\alpha\gamma}}\ \underbrace{{}^{HH}Y^{\overline{\gamma}}}_{-y^{\varepsilon}\Gamma^{\gamma}_{\varepsilon\,\beta}Y^{\beta}}\ ) \\
&\quad + {}^{HH}\ \widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\left(-y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\,\phi}Y^{\phi}\right) + \underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}}_{0}{}^{HH}Y^{c} + \underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}{}^{HH}Y^{\gamma}}_{\Gamma^{\beta}_{\alpha\sigma}Y^{\sigma}} + \underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{HH}Y)^{\overline{\gamma}}) \\
&= X^{\alpha}(\partial_{\alpha}\left(-y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\,\phi}Y^{\phi}\right) + y^{\varepsilon}\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma} \\
&\quad - y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}Y^{\gamma} - \Gamma^{\beta}_{\alpha\gamma}\left(-y^{\varepsilon}\Gamma^{\gamma}_{\varepsilon\,\beta}Y^{\beta}\right)) \\
&\quad + \left(-y^{\varepsilon}\Gamma^{\alpha}_{\varepsilon\,\phi}Y^{\phi}\right)(\partial_{\overline{\alpha}}\left(-y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\,\sigma}Y^{\sigma}\right) + \Gamma^{\beta}_{\alpha\sigma}Y^{\sigma}) \\
&= X^{\alpha}((-\partial_{\alpha}\Gamma^{\beta}_{\varepsilon\,\phi})y^{\varepsilon}Y^{\phi} - y^{\varepsilon}\Gamma^{\beta}_{\varepsilon\,\phi}\left(\partial_{\alpha}Y^{\phi}\right) + \left(\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}\right)y^{\varepsilon}Y^{\gamma} \\
&\quad - y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}Y^{\gamma} - \Gamma^{\beta}_{\alpha\gamma}\Gamma^{\gamma}_{\varepsilon\sigma}y^{\varepsilon}Y^{\sigma}) + \Gamma^{\alpha}_{\varepsilon\,\phi}\Gamma^{\beta}_{\varepsilon\,\sigma}y^{\varepsilon}X^{\phi}Y^{\sigma} - \Gamma^{\alpha}_{\varepsilon\,\phi}\Gamma^{\beta}_{\alpha\sigma}y^{\varepsilon}X^{\phi}Y^{\sigma} \\
&= X^{\alpha}Y^{\phi}y^{\varepsilon}(-\partial_{\alpha}\Gamma^{\beta}_{\varepsilon\,\phi} + \partial_{\varepsilon}\Gamma^{\beta}_{\alpha\phi} - \Gamma^{\beta}_{\alpha\sigma}\Gamma^{\sigma}_{\varepsilon\,\phi} + \Gamma^{\beta}_{\varepsilon\,\sigma}\Gamma^{\sigma}_{\varepsilon\,\phi}) \\
&\quad - y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\phi}X^{\alpha}Y^{\phi} - \Gamma^{\beta}_{\varepsilon\,\sigma}\Gamma^{\sigma}_{\alpha\phi}X^{\alpha}Y^{\phi}y^{\varepsilon} - \Gamma^{\beta}_{\varepsilon\,\phi}y^{\varepsilon}X^{\alpha}\partial_{\alpha}Y^{\phi} \\
&= y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\phi}X^{\alpha}Y^{\phi} - y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\phi}X^{\alpha}Y^{\phi} - \Gamma^{\beta}_{\varepsilon\,\sigma}\Gamma^{\sigma}_{\alpha\phi}X^{\alpha}Y^{\phi}y^{\varepsilon} - \Gamma^{\beta}_{\varepsilon\,\phi}y^{\varepsilon}X^{\alpha}\partial_{\alpha}Y^{\phi} \\
&= {}^{HH}\left(\nabla_{X}Y\right)^{\overline{\beta}}
\end{aligned}
$$

by virtue of (2.6) and (4.3). Thus, we have $\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y})\right) = {}^{HH}(\nabla_X Y)$. □

Theorem 4.5. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X \in \Im_0^1(B_m)$ and $Y \in \Im_0^1(B_m)$, respectively. We have:*

$$ {}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) - \gamma(R(\ ,X)Y). $$

Proof. Using (*iv*) of Theorem 3.1 and Theorem 4.4, we have for any $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$

$$ \begin{aligned} {}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) &= {}^{HH}(\nabla_X Y) + \gamma(R(\ ,X)Y) \\ {}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) &= {}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) + \gamma(R(\ ,X)Y) \\ {}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) &= {}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) - \gamma(R(\ ,X)Y). \end{aligned} $$

Thus we have the Theorem 4.5. □

From (4.1) and (4.2), or from (4.3), we have:

Theorem 4.6. *The complete lift ${}^{cc}\nabla$ and the horizontal lift ${}^{HH}\nabla$ of a projectable linear connection ∇ in B_m coincide, if and only if ∇ is of zero curvature.*

Theorem 4.7. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X \in \Im_0^1(B_m)$ and $Y \in \Im_0^1(B_m)$, respectively. We have:*

$$ {}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y}) = {}^{cc}(\nabla_X Y) - \gamma(R(\ ,X)Y). $$

Proof. If $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$ and $\begin{pmatrix} \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^b \\ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^\beta \\ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^{\overline{\beta}} \end{pmatrix}$ are the components of $\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(B_m)$, then we find

$$ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^J = {}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{cc}\widetilde{Y})^J + {}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{cc}\widetilde{Y})^J + {}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{cc}\widetilde{Y})^J. $$

As the first coordinate, if $J = b$, we obtain

$$ \begin{aligned} \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^b &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{cc}\widetilde{Y})^b + {}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{cc}\widetilde{Y})^b + {}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{cc}\widetilde{Y})^b \\ &= X^a(\underbrace{\partial_a Y^b}_{0} + \underbrace{{}^{HH}\Gamma^b_{ac}}_{0}({}^{cc}\widetilde{Y})^c + \underbrace{{}^{HH}\Gamma^b_{a\gamma}}_{0}({}^{cc}\widetilde{Y})^\gamma + \underbrace{{}^{HH}\Gamma^b_{a\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}}) \\ &\quad + X^\alpha(\partial_\alpha Y^b + \underbrace{{}^{HH}\Gamma^b_{\alpha c}}_{0}({}^{cc}\widetilde{Y})^c + \underbrace{{}^{HH}\Gamma^b_{\alpha\gamma}}_{\Gamma^b_{\alpha\gamma}}\underbrace{{}^{cc}Y^\gamma}_{Y^\gamma} + \underbrace{{}^{HH}\Gamma^b_{\alpha\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}}) \\ &\quad + {}^{cc}X^{\overline{\alpha}}(\underbrace{\partial_{\overline{\alpha}} Y^b}_{0} + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}c}}_{0}({}^{cc}\widetilde{Y})^c + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}\gamma}}_{0}({}^{cc}\widetilde{Y})^\gamma + \underbrace{{}^{HH}\Gamma^b_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}}) \\ &= X^\alpha\left(\partial_\alpha Y^b + \Gamma^b_{\alpha\gamma}Y^\gamma\right) \\ &= {}^{cc}(\nabla_X Y)^b \end{aligned} $$

by virtue of (2.4), (2.5) and (4.3). As the second coordinate, if $J = \beta$, we obtain

$$ \begin{aligned} \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^\beta &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_a({}^{cc}\widetilde{Y})^\beta + {}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_\alpha({}^{cc}\widetilde{Y})^\beta + {}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{cc}\widetilde{Y})^\beta \\ &= X^a(\underbrace{\partial_a Y^\beta}_{0} + \underbrace{{}^{HH}\Gamma^\beta_{ac}}_{0}({}^{cc}\widetilde{Y})^c + \underbrace{{}^{HH}\Gamma^\beta_{a\gamma}}_{0}({}^{cc}\widetilde{Y})^\gamma + \underbrace{{}^{HH}\Gamma^\beta_{a\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}}) \end{aligned} $$

$$
\begin{aligned}
&+X^{\alpha}(\partial_{\alpha}Y^{\beta}+\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha c}}_{0}({}^{cc}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\gamma}}_{\Gamma^{\beta}_{\alpha\gamma}}\underbrace{{}^{cc}Y^{\gamma}}_{Y^{\gamma}}+\underbrace{{}^{HH}\Gamma^{\beta}_{\alpha\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}})\\
&+{}^{cc}X^{\overline{\alpha}}(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}c}}_{0}({}^{cc}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\gamma}}_{0}({}^{cc}\widetilde{Y})^{\gamma}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}})\\
=\;& X^{\alpha}\left(\partial_{\alpha}Y^{\beta}+\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}\right)\\
=\;& {}^{cc}\left(\nabla_{X}Y\right)^{\beta}
\end{aligned}
$$

by virtue of (2.4), (2.5) and (4.3). As the third coordinate, if $J=\overline{\beta}$, then we obtain

$$
\begin{aligned}
\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})\right)^{\overline{\beta}} =\;& {}^{cc}\widetilde{X}^{a\,HH}\nabla_{a}({}^{cc}\widetilde{Y})^{\overline{\beta}}+{}^{cc}\,\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{cc}\widetilde{Y})^{\overline{\beta}}+{}^{cc}\,\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{cc}\widetilde{Y})^{\overline{\beta}}\\
=\;& X^{a}(\underbrace{\partial_{a}({}^{cc}\widetilde{Y})^{\overline{\beta}}}_{0}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{ac}}_{0}({}^{cc}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{a\gamma}}_{0}({}^{cc}\widetilde{Y})^{\gamma}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{a\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}})\\
&+X^{\alpha}(\partial^{cc}_{\alpha}Y^{\overline{\beta}}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha c}}_{0}({}^{cc}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\gamma}}_{y^{\varepsilon}\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}-y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}}\underbrace{{}^{cc}Y^{\gamma}}_{Y^{\gamma}}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\alpha\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}})\\
&+y^{\varepsilon}\partial_{\varepsilon}X^{\alpha}(\partial_{\overline{\alpha}}\left(y^{\varepsilon}\partial_{\varepsilon}Y^{\beta}\right)+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}c}}_{0}({}^{cc}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}}_{\Gamma^{\beta}_{\alpha\gamma}}\underbrace{{}^{cc}Y^{\gamma}}_{Y^{\gamma}}\\
&+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\overline{\gamma}}}_{0}({}^{cc}\widetilde{Y})^{\overline{\gamma}})\\
=\;& y^{\varepsilon}\partial_{\varepsilon}X^{\alpha}(\partial_{\alpha}Y^{\beta})+y^{\varepsilon}\partial_{\varepsilon}X^{\alpha}\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}+y^{\varepsilon}X^{\alpha}\partial_{\alpha}\partial_{\varepsilon}Y^{\beta}\\
&-y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}X^{\alpha}Y^{\gamma}+X^{\alpha}(y^{\varepsilon}\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}Y^{\gamma}+y^{\varepsilon}X^{\alpha}\Gamma^{\beta}_{\alpha\gamma}\partial_{\varepsilon}Y^{\gamma})\\
=\;& {}^{cc}\left(\nabla_{X}Y\right)^{\overline{\beta}}-y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}X^{\alpha}Y^{\gamma}
\end{aligned}
$$

by virtue of (2.4), (2.5) and (4.3). Therewithal, we know that ${}^{cc}\left(\nabla_{X}Y\right)-\gamma(R(\;,X)Y)$ have the components

$$
{}^{cc}\left(\nabla_{X}Y\right)-\gamma(R(\;,X)Y)=\begin{pmatrix}{}^{cc}\left(\nabla_{X}Y\right)^{b}\\ {}^{cc}\left(\nabla_{X}Y\right)^{\beta}\\ {}^{cc}\left(\nabla_{X}Y\right)^{\overline{\beta}}\end{pmatrix}-\begin{pmatrix}0\\0\\y^{\varepsilon}R^{\beta}_{\varepsilon\alpha\gamma}X^{\alpha}Y^{\gamma}\end{pmatrix}
$$

with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t(B_m)$. Thus, we have ${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y})={}^{cc}\left(\nabla_{X}Y\right)-\gamma(R(\;,X)Y)$ in $t(B_m)$. □

Let there be given a projectable linear connection ∇ and a projectable vector field on M_n with projection $X\in\Im^{1}_{0}(B_m)$. Then the Lie derivative $L_{\widetilde{X}}\nabla$ with respect to $\widetilde{X}$ is, by definition, an element of $\Im^{1}_{2}(B_m)$ such that

$$
(L_{\widetilde{X}}\nabla)(\widetilde{Y},\widetilde{Z})=L_{\widetilde{X}}(\nabla_{\widetilde{Y}}\widetilde{Z})-\nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z})-\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z} \tag{4.4}
$$

for any projectable vector fields $\widetilde{Y},\widetilde{Z}\in\Im^{1}_{0}(M_n)$.

A projectable vector field $\widetilde{X}\in\Im^{1}_{0}(M_n)$ [12] with components $\widetilde{X}=\widetilde{X}^{a}(x^{a},x^{\alpha})\partial_{a}+X^{\alpha}(x^{\alpha})\partial_{\alpha}$ is said to be an infinitesimal linear (resp. affine) transformation ([14, p. 67], [11]) in an $m-$dimensional manifold B_m with projectable linear connection ∇, if $L_{\widetilde{X}}\nabla=0$ (see (4.4)).

Theorem 4.8. *Let ∇ be a projectable linear connection on B_m. Then,*

$$
(L_{{}^{cc}\widetilde{X}}{}^{HH}\nabla)({}^{cc}\widetilde{Y},{}^{cc}\widetilde{Z})={}^{cc}\left((L_{\widetilde{X}}\nabla)({}^{cc}\widetilde{Y},{}^{cc}\widetilde{Z})\right)+\gamma(L_{\widetilde{X}}R)(\;,\widetilde{Y})\widetilde{Z})
$$

for any projectable vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im_0^1(M_n)$. *Where* $R(\ ,X)Y$ *denotes a tensor field* F *of type* $(1,1)$ *in* B_m *such that* $F(Z) = R(Z,X)Y$ *for any* $Z \in \Im_0^1(B_m)$.

Proof. Substituting Theorem 4.7 and $(v), (vi)$ of Theorem 3.1 in (4.4), we have

$$\begin{aligned}
(L_{{}^{cc}\widetilde{X}}{}^{HH}\nabla)({}^{cc}\widetilde{Y}, {}^{cc}\widetilde{Z}) &= L_{{}^{cc}\widetilde{X}}({}^{HH}\nabla_{{}^{cc}\widetilde{Y}}{}^{cc}\widetilde{Z}) - {}^{HH}\nabla_{{}^{cc}\widetilde{Y}}(L_{{}^{cc}\widetilde{X}}{}^{cc}\widetilde{Z}) - {}^{HH}\nabla_{\left[{}^{cc}\widetilde{X},{}^{cc}\widetilde{Y}\right]}{}^{cc}\widetilde{Z} \\
&= L_{{}^{cc}\widetilde{X}}\left[{}^{cc}\nabla_{\widetilde{Y}}\widetilde{Z} - \gamma(R(\ ,\widetilde{Y})\widetilde{Z})\right] \\
&\quad -{}^{HH}\nabla_{{}^{cc}\widetilde{Y}}{}^{cc}(L_{\widetilde{X}}\widetilde{Z}) - {}^{HH}\nabla_{{}^{cc}\left[\widetilde{X},\widetilde{Y}\right]}{}^{cc}\widetilde{Z} \\
&= \left[{}^{cc}\widetilde{X}, {}^{cc}\nabla_{\widetilde{X}}\widetilde{Y}\right] - \left[{}^{cc}\widetilde{X}, \gamma(R(\ ,\widetilde{Y})\widetilde{Z})\right] \\
&\quad -{}^{cc}(\nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z})) + \gamma(R(\ ,\widetilde{Y})L_{\widetilde{X}}\widetilde{Z}) \\
&\quad -{}^{cc}\left(\nabla_{\left[\widetilde{X},\widetilde{Y}\right]}\widetilde{Z}\right) + \gamma R(\ ,\left[\widetilde{X},\widetilde{Y}\right])\widetilde{Z}) \\
&= {}^{cc}\left(L_{\widetilde{X}}\nabla_{\widetilde{X}}\widetilde{Y}\right) - {}^{cc}(\nabla_{\widetilde{Y}}(L_{\widetilde{X}}\widetilde{Z})) - {}^{cc}(\nabla_{\left[\widetilde{X},\widetilde{Y}\right]}\widetilde{Z}) - \gamma(L_{\widetilde{X}}R(\ ,\widetilde{Y})\widetilde{Z}) \\
&\quad +\gamma(R(\ ,\widetilde{Y})L_{\widetilde{X}}\widetilde{Z}) + \gamma(R(\ ,L_{\widetilde{X}}\widetilde{Y})\widetilde{Z}) \\
&= {}^{cc}\left(L_{\widetilde{X}}\nabla\right)\left({}^{cc}\widetilde{Y}, {}^{cc}\widetilde{Z}\right) + \gamma(-L_{\widetilde{X}}R(\ ,\widetilde{Y})\widetilde{Z} \\
&\quad +R(\ ,\widetilde{Y})L_{\widetilde{X}}\widetilde{Z} + R(\ ,L_{\widetilde{X}}\widetilde{Y})\widetilde{Z}) \\
&= {}^{cc}\left(L_{\widetilde{X}}\nabla\right)\left({}^{cc}\widetilde{Y}, {}^{cc}\widetilde{Z}\right) + \gamma(L_{\widetilde{X}}R)(\ ,\widetilde{Y},\widetilde{Z}),
\end{aligned}$$

which is the proof of Theorem 4.8. □

From Theorem 4.8, we have

Theorem 4.9. *If* X *is an infinitesimal automorphism with respect to* F *[4], then* ${}^{cc}\widetilde{X}$ *is an infinitesimal linear transformation of* $t(B_m)$ *with* ${}^{HH}\nabla$.

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