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Helmholtz Equation And WKB Approximation In The Tidal Wave Propagation*

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ABSTRACT

The homoegneous radial equation of atmospheric tides has been solved numerically, throuh its analogy with the Helmholtz equation, adoptig a realistic temperature structure below 110 km. Insight intto the properties of the media, a linear law for the variable coefficent is assumed and the relationships between exact solutions of the equation and the WKB approximations are therefore illustrated. The analysis has been carried out for two types of wave Propagation; the oscillatory and the trapped modes of the diurnal and semidiurnal oscillations.

INTRODUCTION

The subject of internal waves in media, such as the atmosphere and oceans, is one of the oldest in fluid dynamics [1]. Atmospheric tides–which may be looked upon as global internal gravity waves resulting from a particular excitation have, therefore, attracted the attention of many investigators, for the last century [2–4].

In the tidal theory [5], the vertical structure of the tidal wave propagation is expressed in terms of a wave function, which is a solution of a linear, inhomogeneous, second order differential equation, with variable coefficient. The characteristics of the vertical propagation of the free oscillations are obtained, therefore, through the solutions of the reduced homogeneous wave equation. This equation shows analogy with the Helmholtz equation, which is a typical one for oscillating

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systems in general. It is, sometimes, argued [6] that once a problem is reduced to this form, it has been solved. This is the line of approach of the present study.

As a boundary value problem, solutions to the reduced homogeneous wave equation are evaluated, for the atmospheric regions below 110 km. The validity of the WKB approximations to the solutions is also investigated, for two types of tidal wave propagation, namely the oscilltory and trapped modes of the diurnal and semidiurnal oscillations.

3. THE VERTICAL WAVE EQUATION

The atmosphere has an infinite number of modes of oscillations, which are excited to varying degrees by the applied tidal force. The latter can be resolved, therefore, into a series of components $J_n(x)$; each of which excites a distinct mode (with number n) of oscillation. For a given mode, the vertical structure–of thermally excited–fields is governed by the solution of the vertical wave equation [7],

$$\frac{d^2y_n}{dx^2} + \mu_n{}^2\,(x)y_n = \frac{k}{\gamma g h_n}\,J_n(x)e^{-x/2} \qquad \ldots.(1)$$

where x, the reduced height, is defined by:

$$x = \int_0^Z \frac{dZ'}{H(Z')} \qquad \ldots.(2)$$

A realistic temperature profile, based on the best available collection of data [8,9]–as relevant to 45°N summer, is utilized in the present analysis to evaluate x. The extracted scale height $H(Z)(=RT(Z)/g)$ is found to be represented fairly accurately by a fourth–degree polynomial in the region ($Z<100$ km); (Fig. la–dotted curve). Above that altitude, a constant positive temperature gradient is assumed, i.e. dH/dZ = Const. (= 0.25). The approximated H(x) profile is shown in Fig. la (solid curve).

$\mu_n(x)$ (Eq. 1) is usually defined as the x–component of the wavenumber; for a given mode (of equivalent depth h_n), $\mu_n(x)$ is solely dependent on the temperature structure in the form:

$$\mu_n^2(x) = -\tfrac{1}{4} + \frac{1}{h_n}(kH + dH/dx) \qquad \ldots.(3)$$

where $k = (\gamma-1)/\gamma$; γ is the ratio of specific heats of air.

We are concerned with the most important contributing modes of the migrating solar diurnal (s = 1) and semidiurnal (s = 2) oscillations; s is the longitudinal wavenumber; those are (1,–2), (1,1) and (2,2), respectively. h_n values for these modes have been previously obtained [10], through the solution of Laplace's tidal equation. Profiles of the computed values of $\mu^2_n(x)$ for these modes are presented in Fig. 1b.

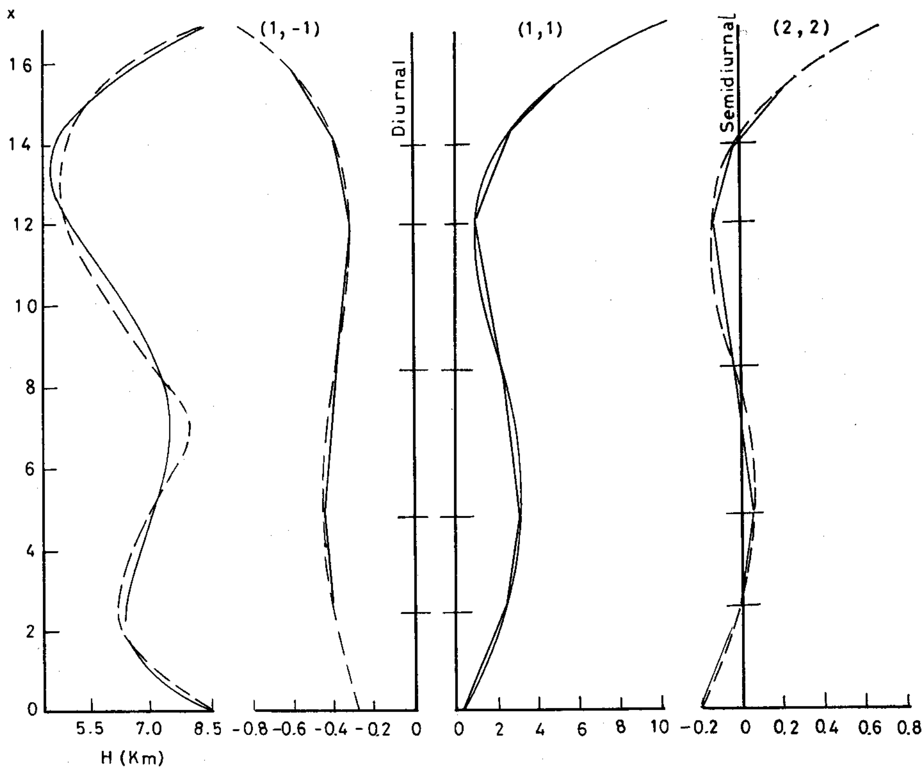

Fig. (1a) Reference atmosphere

Fig. (1b) Refractive index $[\mu^2(x)]$ of the atmosphere to Tidal oscillations.

3. SOLUTIONS TO THE HOMOGENEOUS WAVE EQUATION

The general solution to Eq. (1) can be expressed as the sum of a complementary function and a particular integral; the former is the general solution to the reduced homogeneous equation:

$$\frac{d^2y_n}{dx^2} + \mu^2{}_n(x)y_n = 0 \qquad \ldots.(4)$$

which is analogous to Helmholtz equation [6]; with the variable coefficient depending solely upon T(x) (Eq. 3).

By inspection, the atmosphere below 100 km ($x \leq 15.6$) can be divided to equally spaced subregions, for each of which the variable coefficient $\mu^2{}_n(x)$ can be approximated by a linear law in the form:

$$\mu^2{}_n(x) = P_n x + Q_n \qquad \ldots.(5)$$

The linear representation (5) is shown in Fig. 1b. For all modes considered, it is obvious that the atmosphere, below 100 km, is divided into three main regions–with different characteristics to the propagation of the tidal waves–separated at levels $x = 4.8$ and 9.6.

In order to facilitate the discussion, it will be assumed that one mode is excited at a time, and drop the suffix n–thereafter–for simplicity in writing. Introducing a new independent variable:

$$s = \int^x \mu dx = \frac{2}{3p}\mu^3 \; ; \; \mu^2 > 0 \qquad \ldots.(6)$$

or

$$\sigma = \int^x \lambda dx = \frac{2}{3p}\lambda^3; \; \mu^2 < 0 \; (= -\lambda^2) \qquad \ldots.(7)$$

Helmholtz equation, then assumes the form of Stoke's equation, [6],

$$\frac{d^2y}{ds^2} + sy = 0 \qquad \ldots.(8)$$

Thus, the solutions to the homogeneous equation (4) are:

$$y(x) = \mu \; [AJ_{1/3}(s) + BJ_{-1/3}(s)]; \; \mu^2 > 0 \qquad \ldots.(9)$$

or

$$y(x) = \lambda \; [-AI_{1/3}(\sigma) + BI_{-1/3}(0)], \; \mu^2 < 0 \; (= -\lambda^2) \qquad \ldots.(10)$$

where $J_{\pm 1/3}$ and $I_{\pm 1/3}$ are the Bessel functions of the real and imaginary arguments, respectively. A(= a + ib) and B (= c + id) are two integration constants.

At heights corresponding to $15.6 \leq x \leq 16.8$, $\mu^2(x)$ has been adjusted to meet smoothly the temperature model dH / dZ = b', i.e.

$$\mu^2(x) = Ce^{b'x} - b' \quad \ldots.(11)$$

where $C = (k + b')\, H(x^*)e^{-bx^*} / h$, $x^* = 15.6$, $b' = 0.25$

let $$s(x) = i\,\sigma(x) = \frac{2}{b'} \sqrt{C}\; e^{b'x/2} \quad \ldots.(12)$$

Thus the homogeneous wave equation (4) assumes the Bessel differential equation of the form:

$$\frac{d^2y}{ds^2} + \frac{1}{s}\,\frac{dy}{ds} + \left(1 - \frac{1}{b'^2 s^2}\right) y = 0; \; \mu^2 > 0 \quad \ldots(13)$$

or Bessel modified differential equation for $s^2(x) < 0$ (negative modes). In either case, the general solution to Eq. (4) is:

$$y(x) = AJ_{1/b'}(s) + BY_{1/b'}(s); \quad \mu^2 > 0 \quad \ldots.(14)$$

$$\text{or} \quad y(x) = AI_{1/b'}(\sigma) + BK_{1/b'}(\sigma); \quad \mu^2 < 0 \quad \ldots.(15)$$

$J_{1/b'}$ and $Y_{1/b'}$ ($I_{1/b'}$ and $K_{1/b'}$ are Bessel (modified Bessel) functions of the first and second kinds, respectively, of order 1/b' (= 4 in the present study).

Therefore, the general solution to the reduced homogeneus equation (4) can be represented as:

$$y(x) = Ay_1(x) + By_2(x) \quad \ldots.(16)$$

Bessel functions have been obtained at all levels concerned x = 0 (0.1) 16.8, for the above mentioned modes, and consequently the linearly independent solutions y_1 and y_2 are avaluated. It remains to obtain the integration constants A and B for each one of the subregions.

The linearized tidal theory requires that contact should be maintained at all times at the boundaries between the atmospheric regions, i.e. it requires continuity in y and dy / dx. Thus the arbitrary constants A & B, for all the subregions are related.

4. BOUNDARY CONDITIONS

As a lower boundary condition, it is generally assumed [7] that the vertical component of velocity must vanish at the surface, this implies that:

$$\left.\frac{dy}{dx}\right|_{x=0} = \left(\frac{1}{2} - \frac{H(o)}{h}\right) y(o) \qquad \ldots.(17)$$

On using (16), the condition (17) gives the ratio A / B ($= \alpha$ say).

At high levels, $Z > 110$ km, it is required [11], first, that y be bounded and secondly, that the flow of energy be in the upward direction. The boundness of solution is met for $\mu^2(x) > O$ (Eq. 14), but in case of $\mu^2(x) < O$ (Eq. 15) the growing exponential of $I_{1/b}(\sigma)$ in the upper–most region is dropped, in order to avoid the paradox of infinite energy densities.

5. CHARACTERISTICS OF THE FREE TIDAL OSCILLATIONS:

The general solution (16) of the homogeneous equation (4) may then be written in a complex form, representing a wave solution, as:

$$y(x) = a'\gamma(x)\ e^{i\,\varnothing(x)} \qquad \ldots.(18)$$

constant B is incorporated into the phase, and the constant a′, and $\gamma(x) = (A / B)y_1(x) + y_2(x)$ with A / B given by the use of the lower boundary condition (17). $\gamma(x)$ results, thus obtained for the above mentioned modes, are presented in table 1 at x=0 (1.2) 16.8; also included in the table are the P′s values (Eq. 5) for each subregion. The δ column will be dealt with in the next section. The solutions of the homogeneous equation, reveal the following features:

(i) The diurnal mode (1,–2): $\mu^2(x)<O$ region extends to the top of the domain, and is not much affected by the vertical temperature structure; $P \ll 1$ and consequently $\sigma \gg 1$ (Eq. 7). In such cases $I_{\pm 1/3}(\sigma)$ assume the asymptotic forms and to satisfy the upper boundary condition one must have, in theory [6], A = B; A / B; = 0.8898 in the present study. At x = 4.8, P = –0.084 for the region below and P = 0.021 above, leading to σ = 16.0 and 63.8, respectively. For $\sigma \gg 1$, solution (18) reduces to the asymptotic form solution:

$$y(x) \sim \frac{3}{2}\, B \sqrt{\frac{P}{\pi\lambda}}\ e^{\pm\sigma} \qquad \ldots.(19)$$

This explains the characteristic behaviour of the solution obtained for this mode (table 1); the level (x = 4.8, Z∼33 km) clearly

Tabe 1. Characteristics of the tidal oscillations

A / B	Diurnal (1,–2) 0.8898			Diurnal (1,1) 6.0103			Semidiurnal (2,2) 0.6882		
x	P	y(x)	δ	P	y(x)	δ	P	y(x)	δ
0		0.0002			–0.5576			–0.0001	
	–0.050		0.12	0.870		0.24	0.077		1.11
1.2		0.0773			–1.3283			–0.0328	
	–0.040		0.08	0.707		0.10	0.063		—
2.4		0.2115			–0.3307			–0.0174	
	–0.027		0.05	0.475		0.05	0.042		3.32
3.6		0.4814			–2.2975			–5.7638	
	–0.012		0.02	0.210		0.02	0.019		0.83
4.8		1.0658			–1.3467			–1.8565	
	0.003		0.01	–0.054		0.01	–0.005		0.24
6.0		0.4806			1.8110			–2.8012	
	0.016		0.03	–0.281		0.03	–0.025		3.33
7.2		0.2169			–1.8552			–22.6119	
	0.026		0.05	–0.437		0.07	–0.039		3.33
8.4		0.0967			–0.4755			0.0078	
	0.028		0.07	–0.487		0.12	–0.043		0.83
9.6		0.0424			13.3249			0.0596	
	0.023		0.06	–0.398		0.16	–0.035		0.42
10.8		0.0194			13.4321			0.1319	
	0.007		0.02	–0.127		0.06	–0.011		0.11
12.0		0.0098			12.6591			0.2026	
	–0.020		0.05	0.353		0.10	0.031		0.48
13.2		0.0209			17.6114			0.1056	
	–0.062		0.12	1.079		0.12	0.096		—
14.4		0.0760			–83.1085			–290.932	
	–0.119		0.14	2.086		0.09	0.186		0.83
15.6		0.3356			–213.9691			–1.0901	
16.8		0.8426			10.8513			–1.8158	

separates two different regions of the atmospheric refractive indices (Fig. 1a). Finally, the constancy in phase, incorporated in B suggests the trapping of energy; a feature characteristic to the negative modes, whose discovery was the most important contribution to theory of atmospheric tides [12].

(ii) The diurnal mode (1,1): The function y(x) is in general oscillatory, with a wave length $\sim$ 25 km, which is given by the reciprocal of μ Once again, for large s, resulting from small p (Eq. 6). one may use the asymptotic forms [6] for $J_{\pm 1/3}(s)$ to give:

$$y(x) \simeq 3B \sqrt{\frac{P}{\pi\mu}} \text{ Cos } (s - \pi / 4) \qquad \ldots.(20)$$

as revealed by the results for y(x) in table 1. The oscillatory behaviour is not that pronounced, however, in the region just above the temperature maximum ($9.6 < x < 13.2$). This, no doubt, can account for the weakness and great irregularities observed in the solar diurnal oscillation [13], due to trapping or interference.

(iii) The semidiurnal mode (2,2): This mode is a transition mode in the sense that $\mu^2 < 0$ ($x < 2.8$, $7.7 \leq x \leq 14.2$) and $\mu^2 > 0$ elsewhere. Therefore, y(x) depends significantly on the temperature distribution, a fact which was central to the old resonance theory [14]. The vertical wavelength of this mode, in general, exceeds 100 km–in the top layer –and there is an appreciable region where the mode is evanescent and barely grows (between 50 and 80 km).

6. WKB APPROXIMATION IN THE TIDAL WAVE PROPAGATION:

Substituting the complex form solution (18) into Eq. (4) and separation of the real & imaginary parts gives:

$$\frac{d^2Y}{dx^2} - Y \left[\left(\frac{d\varphi}{dx}\right)^2 - \mu^2\right] = 0 \qquad \ldots.(21)$$

$$\& \quad Y \frac{d^2\varphi}{dx^2} + 2 \left(\frac{d\varphi}{dx}\right)\left(\frac{dY}{dx}\right) = 0 \qquad \ldots.(22)$$

Eq. (22) is immediately soluble by quadrature:

$$Y \left(\frac{d\varphi}{dx}\right)^{\frac{1}{2}} = C \qquad \ldots.(23)$$

C being a constant of integration

Applied to the problem at hand, WKB approximation [6] consists in assuming that the first term in (21) is negligible, i.e.

$$\left(\frac{d\varnothing}{dx}\right)^2 \simeq \mu^2 \qquad \ldots.(24)$$

This is equivalent to the assumption that changes in $\mu(x)$ become small enough over a wavelength in the verticle, i.e.

$$\left| \frac{1}{\mu} \frac{d}{dx} \ln \mu \right| \ll 1 \qquad \ldots.(25)$$

The WKB parameter δ $\left(= \left| \frac{1}{\mu} \frac{d}{dx} \ln \mu \right| \right)$ is given in table 1,

for the above mentioned modes, from which it is clear that the WKB approximation is valid for the whole range in case of the diurnal modes. In case of the (2,2) mode, we get three planes at which $\mu^2 = 0$, i.e.

$$\left| \frac{1}{\mu} \frac{d}{dx} \ln \mu \right| \longrightarrow \infty \qquad \ldots.(26)$$

at $x = 2.4$, 7.2 & 14.4– corresponding to the turning points of the Helmholtz equation, and WKB approximation breaks down.

If there are abrupt changes in μ, a second approximation may also be used; this is to assume μ uniform except at a discontinity surface at some height. This case is quite analogous to the reflection of electromagnetic waves at the interface between media of different index of refraction [15].

In vitrue of (23) and (24), Eq. (19) thus gives:

$$y(x) = \frac{C}{\sqrt{\mu}} e^{\pm i\mu x} \qquad \ldots.(27)$$

Let us consider a tidal wave of amplitude $C_i / \sqrt{\mu_1}$, having a downward phase propagation but an upward energy propagation to the interface from below, a reflected awave $C_r / \sqrt{\mu_1}$ in the lower region and a transmitted wave $C_t / \sqrt{\mu_2}$ in the upper region. Without loss in generality, the interface may be taken to be at x = O and numbers 1,2 indicate quantities corresponding to media 1(x<O) and 2(x>O). If discontinuities in pressure and vertical velocity are to be avoided, both y and dy / dx must be continuous across the interface. Therefore we get:

$$C_r / C_i = \frac{\mu_2 / \mu_1 - 1}{\mu_2 / \mu_1 + 1} \qquad \ldots .(28)$$

At x = 15.6, it has been found that $C_r / C_i = 0.10$ and 0.15 for the oscillatory diurnal and semidiurnal modes ($\mu^2 > O$), respectively. As the energy flux is proportional to the square of the wave magnitude, 1–2 % of the energy of these modes will be reflected and the rest will therefore be transmitted to the thermosphere. The possibility of significant tidal heating of the thermosphere, by upward propagating energy fluxes, has been previously investigated [13, 14].

7. CONCLUSION

To conclude, we have attempted, in the present study, to describe the behaviour of free tidal waves, under naturally occurring conditions in static, stratified fluids–through the analogy of the homogeneous wave equation with Helmholtz equation. A main parameter that governs the tidal wave propagation is the refractive index of the atmosphere, which for a given mode of oscillation is solely dependent on the temperature structure. Therefore, the assumption of isothermal atmosphere, previously adopted in order to render the mathematical treatment of the tidal equations more tractable, is rather a drastic one.

The merit of the present study is to represent the general solution of the reduced homogeneous equation analytically. This will be used in further inverstigation to obtain the general solution of the inhomogeneous vertical wave equation.

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