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Cesâro Difference Sequence Spaces And Related Matrix Transformations

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Cesáro Difference Sequence Spaces And Related Matrix Transformations

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SUMMARY

In this paper, we have defined Cesáro difference sequence spaces C_p, $1 \leq p < \infty$, and C_∞, namely,

$$C_p = \left\{ x = (x_k): \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p < \infty, 1 \leq p < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_k): \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| < \infty, n \geq 1 \right\},$$

respectively, and investigated some properties of these spaces. Further, we have determined the matrices of classes (C_p, E), (C_∞, E) and (E, C_p), (E, C_∞), where E denotes one of the sequence speces l_∞ and c namely the linear space of bounded sequences and convergent sequences, respectively.

1. INTRODUCTION

The Cesáro sequence spaces

$$ces_p = \left\{ x = (x_k): \| x \|_p = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty, 1 \leq p < \infty \right\}$$

and

$$ces_\infty = \left\{ x = (x_k): \| x \|_\infty = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$$

have been introduced by J.S. Shiue [4], where $x = (x_k)$ is a sequence of real numbers. And it has been shown that the inclusion $l_p \subset ces_p$ is strict for $1 < p < \infty$, althought it does not hold for $p = 1$, where

$$l_p = \left\{ x = (x_k): \sum_{k=1}^{\infty} |x_k|^p < \infty, 1 \leq p < \infty \right\}$$

Furthermore, the Cesáro sequence space X_p of non-absolute type is defined by P.N.Ng and P.Y.Lee, in [2], as follows:

$$X_p = \left\{ x = (x_k): \|x\|_p = \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p \right)^{1/p} < \infty, 1 \leq p < \infty \right\}$$

and

$$X_\infty = \left\{ x = (x_k): \|x\|_\infty = \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| < \infty, n \geq 1 \right\}.$$

Moreover, it has been shown that the space X_p, $1 \leq p \leq \infty$, is a Banach space of non-absolute type and the inclusion $ces_p \subset X_p$, $1 \leq p \leq \infty$, is strict, [2]. And the matrix transformations on Cesáro sequence spaces of a non-absolute type have been introduced by P.N.Ng [3].

The main purpose of this paper, is to define the Cesáro difference sequence spaces and to investigate some properties of these spaces. Then we will determine some of the related matrix transformations.

2. DEFINITIONS

Let $x = (x_k)$ be a sequence of complex numbers and $\triangle x_k = x_k - x_{k+1}$, $(k = 1, 2, \ldots)$. Let us define

$$C_p = \left\{ x = (x_k): \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p < \infty, 1 \leq p < \infty \right\}$$

and

$$C_\infty = \left\{ x = (x_k): \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty, n \geq 1 \right\}.$$

Obviously, C_p $(1 \leq p \leq \infty)$ is a linear space with the usual operations.

Now let us define

$$(1) \quad \|x\|_p = |x_1| + \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \right)^{1/p}, 1 \leq \; < \infty,$$

(2) $\|x\|_\infty = |x_1| + \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|$.

Therefore $x \in C_p$ if and only if $\|x\|_p < \infty$, $1 \leq p \leq \infty$,

The following theorem is straightforward.

THEOREM 2.1. The spaces C_p, $1 \leq p < \infty$, and C_∞ are Banach spaces with the norms (1) and (2), respectively.

3. INCLUSION THEOREMS

In this section, we give some inclusion theorems between related sequence spaces.

THEOREM. 3.1. If $1 \leq p < q$, then $C_p \subseteq C_q$.

Proof. The inequality

$$\left(\sum_{k=1}^{n} |a_k|^q \right)^{1/q} \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p}, \ (0 < p < q),$$

gives the proof, [1; p.4].

THEOREM 3.2. The inclusion $X_p \subset C_p$, $1 \leq p \leq \infty$, is strict.

Proof. Let $x = (x_k) \in X_p$, $1 \leq p < \infty$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right| \leq \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| + \left| \frac{1}{n} \sum_{k=1}^{n} x_{k+1} \right|.$$

It is known that, for $1 \leq p < \infty$,

$$|a+b|^p \leq 2^p . (|a|^p + |b|^p).$$

Hence, for $1 \leq p < \infty$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p \leq K \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p + \left| \frac{1}{n} \sum_{k=1}^{n} x_{k+1} \right|^p \right\}$$

where $K = 2^p$. Then, for each positive integer m, we get

$$\sum_{n=1}^{m} \left| \frac{1}{n} \sum_{k=1}^{n} \Delta x_k \right|^p \leq K \left\{ \sum_{n=1}^{m} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p + \sum_{n=1}^{m} \left| \frac{1}{n} \sum_{k=1}^{n} x_{k+1} \right|^p \right\}$$

Now, as $m \to \infty$

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p \leq K \left\{ \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right|^p + \right.$$

$$\left. \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} x_{k+1} \right|^p \right\} < \infty.$$

Thus $X_p \subset C_p$, $(1 \leq p < \infty)$. This inclusion is strict since the sequence $x = (1, 1, \ldots)$, for example, belongs to C_p, but does not belong to X_p for $1 \leq p < \infty$. Similarly, it can be easily shown that $X_\infty \subset C_\infty$. To see that $X_\infty \neq C_\infty$, we define the sequence (x_k) by $x_k = k$, $(k = 1, 2, \ldots)$. Then (x_k) is a member of C_∞, but not of X_∞.

Therefore, for $1 < p < \infty$, the inclusion

$$l_p \subset ces_p \subset X_p \subset C_p$$

is strict.

REMARK. If we define

$$O_p = \left\{ x = (x_k) : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\triangle x_k| \right)^p < \infty,\ 1 \leq p < \infty \right\}$$

$$O_\infty = \left\{ x = (x_k) : \sup_n \frac{1}{n} \sum_{k=1}^{n} |\triangle x_k| < \infty,\ n \geq 1 \right\},$$

then these spaces are normed spaces under the following norms respectively.

$$\|x\|_p = |x_1| + \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |\triangle x_k| \right)^p \right)^{1/p},\ 1 \leq p < \infty,$$

and

$$\|x\|_\infty = |x_1| + \sup_n \frac{1}{n} \sum_{k=1}^{n} |\triangle x_k|.$$

Clearly, $O_p \subset C_p$, $1 \leq p \leq \infty$, with a strict inclusion. Actually, the sequence $(x_k) = ((-1)^k)$, $(k = 1,2, \ldots)$, is an element of C_p, but is not an element of O_p for $1 < p < \infty$. Moreover the sequence (x_k) defined by $x_1 = 0$, $x_k = (-1)^k/k$, $(k = 2,3, \ldots)$, belongs to C_1 but not belongs to O_1. If we define $x_k = (-1)^k \cdot k$, $(k = 1,2, \ldots)$, then $(x_k) \notin O_\infty$ but $(x_k) \in C_\infty$.

On the other hand, it is easily seen that $ces_p \subset O_p$, for $1 \leq p \leq \infty$, and this inclusion is also strict. Note that X_p and O_p, $1 \leq p \leq \infty$, overlap but neither one contains the other.

4. DUAL SPACES

In this paragraph we determine the β- dual (generalized Köthe-Toeplitz dual) of C_p, $1 \leq p \leq \infty$, and obtain some results useful in the characterization of certain matrix transformations.

Now let us define the operator $S: C_p \to C_p$, $x \to Sx = (0, x_2, x_3, \ldots)$, $(1 \leq p \leq \infty)$. It is clear that S is a bounded linear operator on C_p with $\|S\| = 1$. Furthermore,

$$S(C_p) = \{x = (x_k): x \in C_p, x_1 = 0\} \subset C_p$$

is a subspace of C_p, $1 \leq p \leq \infty$.

Now, we can give the following lemma.

LEMMA. 4.1. Let σ be defined on $S(C_p)$ by $\sigma(x) = (\sigma_n(x))$, where

$$\sigma_n(x) = \frac{1}{n} \sum_{k=1}^{n} \triangle x_k = \frac{-x_{n+1}}{n}, \quad (n = 1, 2, \ldots),$$

$(1 \leq p \leq \infty)$. Then σ is an one-to-one bounded linear transformation from $S(C_p)$ onto the sequence space l_p with operator norm 1.

The proof is trivial.

It is well-known that, if X is a sequence space, then

$$X^\beta = \left\{ a = (a_k): \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X \right\}.$$

THEOREM. 4.2. If we set

$D_1 = \{a = (a_k): (k\, a_k) \in l_\infty\}$

$D_2 = \{a = (a_k): (k\, a_k) \in l_q, p^{-1} + q^{-1} = 1, 1 < p < \infty\}$

$D_3 = \{a = (a_k): (k\, a_k) \in l_1\}$, then $(S(C_1))^{\beta} = D_1$ and $(S(C_p))^{\beta} = D_2$ where $1 < p < \infty$, and $(S(C_\infty))^{\beta} = D_3$.

Proof. Suppose that $a = (a_k) \in (S(C_p))^{\beta}$, $1 < p < \infty$. Then $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in S(C_p)$. Hence, for each $x \in S(C_p)$

$$(3) \quad \sum_{k=1}^{\infty} a_k x_k = \sum_{k=2}^{\infty} a_k x_k = \sum_{k=1}^{\infty} x_{k+1} a_{k+1} = - \sum_{k=1}^{\infty} k\, t_k\, a_{k+1}$$

where $t_k = \frac{1}{k} \sum_{i=1}^{k} \triangle x_i = \frac{-x_{k+1}}{k}$; $(k = 1,2,\ldots)$.

Therefore, considering Lemma 4.1 we see that the series in (3) converges for all sequences $t = (t_k)$ belonging to l_p, $1 < p < \infty$. This shows that the sequence $(k\, a_{k+1})$ belongs to l_q, where $p^{-1} + q^{-1} = 1$, so $a \in D_2$.

Conversely, if $a \in D_2$, then

$$(4) \quad \sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} \frac{x_{k+1}}{k}\, k\, a_{k+1}$$

for each $x \in S(C_p)$, $1 < p < \infty$. If we now consider Lemma 4.1 and apply the Hölder inequality, then the series (4) is absolutely convergent for $1 < p < \infty$. Hence $D_2 = (S(C_p))^{\beta}$, $1 < p < \infty$.

Similarly, it can easily be seen that

$(S(C_1))^{\beta} = D_1$ and $(S(C_\infty))^{\beta} = D_3$.

This completes the proof.

Note that $(S(C_p))^{\beta} = (C_p)^{\beta}$, $1 \leq p \leq \infty$.

5. MATRIX TRANSFORMATIONS

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n, k = 1,2,\ldots)$ and X, Y be two subsets of the space of complex sequences, We write formally

$$(5) \quad A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k, \ (n = 1,2,\ldots),$$

and say that the matrix $A = (a_{nk})$ defines a matrix transformations from X into Y and it is denoted by writing $A \in (X, Y)$, if each series in (5) converges and $(A_n(x)) \in Y)$ whenever $(x_k) \in X$. Furthermore, let (X, Y) be the set of all infinite matrices $A = (a_{nk})$ which map the sequence space X into the sequence space Y. We now determine the matrices of classes (C_p, E), $1 < p \leq \infty$, and (E, C_p), $1 \leq p \leq \infty$, where E denotes one of the sequence spaces l_∞, all bounded complex sequences, and c, all convergent complex sequences.

THEOREM. 5.1. $A \in (C_p, E)$, $1 < p \leq \infty$, if and only if

(i) $(a_{n1}) \in E$

(ii) $B \in (l_p, E)$

where $B = (b_{nk}) = (k\, a_{n, k+1})$ for all n, k.

Proof. Necessity: If $A \in (C_p, E)$, $1 < p \leq \infty$, then the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ is convergent for each n and for all $x \in C_p$, and $(A_n(x)) \in E$. Since the sequence $x = (1, 0, 0, \ldots)$ is an element of C_p, $1 < p \leq \infty$, we get $(A_n(x)) = (a_{n1}) \in E$. Further, by Theorem 4.2, the sequence $(ka_{n, k+1})$ is an element of l_q for every n, where $p^{-1} + q^{-1} = 1$. Moreover, for all $x \in S(C_p) \subset C_p$, $1 < p \leq \infty$, and for all n

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k = \sum_{k=2}^{\infty} a_{nk}x_k = \sum_{k=1}^{\infty} a_{n, k+1}\, x_{k+1},$$

so, the series

$$(6)\quad A_n(x) = - \sum_{k=1}^{\infty} k\, a_{n, k+1}\, t_k$$

is convergent for all sequences $t = (t_k)$ belonging to l_p where

$t_k = \frac{-x_{k+1}}{k}$, $(k = 1,2, \ldots)$. Thus $B \in (l_p, E)$, $1 < p \leq \infty$, where

$$B = (b_{nk}) = (k\, a_{n, k+1})$$

for all n, k. This proves the necessity.

Sufficiency: Suppose (i) and (ii) hold. If $x \in C_p$, $1 < p \leq \infty$, then

$$x_k = \begin{cases} x_1, & k = 1 \\ x'_k, & k \geq 2 \end{cases}$$

where $x' = (x'_k) \in S(C_p)$. On the other hand, let us write formally

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k = a_{n1} x_1 + \sum_{k=2}^{\infty} a_{nk} x'_k$$

$$= a_{n1} + \sum_{k=1}^{\infty} a_{n,k+1} x'_{k+1}.$$

So that

$$A_n(x) = a_{n1} x_1 - \sum_{k=1}^{\infty} k\, a_{n,k+1} t_k$$

where $t_k = \frac{-x'_{k+1}}{k}$ and $(t_k) \in l_p$ by Lemma 4.1. Now (i) and (ii) imply together that $A_n(x)$ exists for each $x \in C_p$ and $A \in (C_p, E)$, $1 < p \leq \infty$. Hence the proof is complete.

THEOREM. 5.2. $A \in (E, C_p)$, $1 \leq p \leq \infty$, if and only if

(i) $\sum_{k=1}^{\infty} |a_{nk}| < \infty$, for each n

(ii) $B \in (E, l_p)$

where $B = (b_{ik}) = \frac{1}{i}(a_{1k} - a_{i+1,k})$ for all i, k.

Proof. Sufficiency is trivial.

Necessity. Suppose that $A = (a_{nk})$ maps E into C_p, $(1 \leq p \leq \infty)$. Then the series

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$$

is convergent for each n and for all $x \in E$ and $(A_n(x)) \in C_p$. Since $E^{\beta} = l_1$ for $E = l_\infty$ or c, then we get (i). Furthermore, since $(A_n(x)) \in C_p$,

$$\sum_{i=1}^{\infty} \left| \frac{1}{i} \sum_{n=1}^{i} \triangle A_n(x) \right|^p = \sum_{i=1}^{\infty} \left| \frac{1}{i} (A_1(x) - A_{i+1}(x)) \right|^p < \infty$$

for all $x \in E$ and for $1 \leq p < \infty$. Whereas

$$\frac{1}{i} (A_1(x) - A_{i+1}(x)) = \sum_{k=1}^{\infty} \frac{1}{i} (a_{1k} - a_{i+1,k}) x_k,$$

for $x \in E$. If we now set

$$B_i(x) = \sum_{k=1}^{\infty} \frac{1}{i} (a_{1k} - a_{i+1,k}) x_k,$$

then $(B_i(x)) \in l_p$, $(1 \leq p < \infty)$. So that $B \in (E, l_p)$ where

$$B = (b_{ik}) = \frac{1}{i} (a_{1k} - a_{i+1,k})$$

for all i, k. The case $p = \infty$ it is also obtained in a similar way. Hence the necessity is proved.

ÖZET

Bu çalışmada

$$C_p = \left\{ x = (x_k): \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right|^p < \infty, 1 \leq p < \infty \right\}$$

ve

$$C_\infty = \left\{ x = (x_k): \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \triangle x_k \right| < \infty, n \geq 1 \right\}$$

Cesáro fark dizi uzayları tanımlanmış ve bu uzayların bazı özellikleri incelenmiştir. Ayrıca E, sınırlı diziler uzayı olan l_∞ ve yakınsak diziler uzayı olan c den herhangi birini göstermek üzere $(C_p, E), 1 < p \leq \infty$, ve $(E, C_p), 1 \leq p \leq \infty$, matris sınıfları belirlenmiştir.

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