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by

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Comparison And Oscillation Theorems For Singular Ultrahyperbolic Equations

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SUMMARY

In this study, comparison and oscillation theorems are given for a class of singular ultra-hyperbolic partial differential equations in a domain HxG $\subset E^{m+n}$, where H is the domain bounded by two concentric m-spheres having centers at ξ in E^m and G is a bounded regular domain in E^n. By using a suitable coordinate transformation the considered equation reduces to a singular hyperbolic equation.

1. INTRODUCTION

The Sturmian comparison and oscillation theorems for ordinary differential equations have been extended extensively to partial differential equations of the elliptic type. For example, see Kuks [1], Swanson [2], [3], Diaz and McLaughlin [4], and Kreith and Travis [5], to mention only a few. In [6] by employing Swanson's technique, Dunninger obtained a comparison theorem for parabolic partial differential equations. His result was recently generalized by Chan and Young [7] to time-dependent quasilinear differential systems. However, for partial differential equations of hyperbolic type very little is known. In fact, as far as this author knows, the only comparisoon result known for hyperbolic equations were obtained just recently by Kreith [8], Travis [9], Young [10] and Travis and Young [11].

In [8] Kreith proved comparison and oscillation theorems for solutions of an initial boundary value problem for the damped wave equation in two variables. His results have been recently extended by Travis [9] to the normal hyperbolic equation in n space variables, and

by Young [10] to singular hyperbolic equation in n space variables.
Other oscillation results for solutions of hyperbolic equations have also
been obtained by Kahane [12] under somewhat different conditions
and by Mamoru Narita [13] for solutions of semilinear hyperbolic and
ultrahyperbolic equations.

In this paper, we shall present some comparison and oscillation theo-
rems for a class of singular ultrahyperbolic equations.

Let us consider the pair of singular ultrahyperbolic equations

$$Lu \equiv \sum_{i=1}^{m} \left(u_{x_i x_i} + \frac{\alpha_i}{x_i - \xi_i} u_{x_i} \right) - \sum_{i,j=1}^{n} D_j[a_{ij}(r,y) D_i u] + p(r,y) u = 0 \tag{1}$$

and

$$Mv \equiv \sum_{i=1}^{m} \left(v_{x_i x_i} + \frac{\beta_i}{x_i - \xi_i} v_{x_i} \right) - \sum_{i,j=1}^{n} D_j[b_{ij}(r,y)D_i v] + q(r,y) v = 0 \tag{2}$$

where α_i and β_i are real parameters, $-\infty < \alpha_i < \infty, -\infty < \beta_i < \infty, i =$
$1,\ldots,$ m. As usual $x = (x_1,\ldots,x_m)$ and $y = (y_1,\ldots,y_n)$ denote points
in E^m and E^n, respectively, and D_i denotes partial differentiation with
respect to y_i, $i = 1,\ldots,n$. The notation $|x|$ will be used for the Eu-
clidean norm of $x \in E^m$. It is assumed that the coefficient matrices (a_{ij})
and (b_{ij}) are symmetric, positive definite and of class C^1 while p and q
are simply continuous with the elements $r = |x-\xi|$ and y in a do-
main

$$\Omega_{[\varepsilon,r]} = H_{[\varepsilon,r]} \times G$$

where G is a bounded domain in E^n with smooth boundary and $H_{[\varepsilon,r]}$
is a domain bounded by two concentric m spheres having centers at
$\xi = (\xi_1,\ldots,\xi_m)$, defined by

$$H_{[\varepsilon,r]} = \{x \in E^m: \varepsilon \le |x-\xi| \le \Gamma\}$$

$\varepsilon > 0, \Gamma < \infty$, and $|x-\xi| = r$ is the radial variable. We shall also
use the notations

$$H_{(\varepsilon,r)} = \{x \in E^m: \varepsilon < |x-\xi| < \Gamma\}$$

and

$$S_r = \{x \in E^s: |x-\xi| = r\}$$

By a solution $u(\sqrt{(x_1-\xi_1)^2 + \ldots + (x_m-\xi_m)^2}; y_1,\ldots,y_n) = u(r,y)$ of $Lu=0$

or a solution $v(\sqrt{(x_1-\xi_1)^2 + \cdots + (x_m-\xi_m)^2};y_1, ..., y_n) = v((r,y))$ of $Mv=0$ we shall mean a function of class $C^2(\Omega_{(\varepsilon, r)}) \cap C^1 (\Omega_{[\varepsilon, r]})$.

In section 2 we shall present comparison theorems for the singular ultrahyperbolic equations (1) and (2), using the method of proof given in [10] by Young. In section 3 we shall present oscillation theorems for singular ultrahyperbolic equations using a technique developed by Noussair and Swanson [14] for systems of ordinary differential equations. This technique was applied to hyperbolic equations by Travis [9] and to singular hyperbolic equations by Young [10].

2. Comparison Theorems.

Let us observe that the operators which are on the first parts of L and M, can be expressed in the radial variable $r = |x - \xi|$ as follows,

$$L_1 u \equiv \sum_{i=1}^{m} \left(u_{x_i x_i} + \frac{\alpha_i}{x_i-\xi_i} u_{x_i} \right) = u_{rr} + \frac{m-1 + \sum_{i=1}^{m} \alpha_i}{r} u_r$$

$$M_1 v \equiv \sum_{i=1}^{m} \left(v_{x_i x_i} + \frac{\beta_i}{x_i-\xi_i} v_{x_i} \right) = v_{rr} + \frac{m-1 + \sum_{i=1}^{m} \beta_i}{r} v_r$$

We want to point out that, it will be needed to take $\sum_1^m \alpha_i = \sum_1^m \beta_i$

in order to prove our theorems in this paper. For brevity we define the parameter $\varnothing$ by

$$\varnothing = \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i$$

We treat first the case when L and M have the same principal parts, that is, $a_{ij} = b_{ij}$, for $i, j = 1,...,n$. Let us consider the singular boundary value problems

$$L_1 u - \sum_{i,j=1}^{n} D_j(a_{ij} D_i u) + pu = 0 \text{ in } \Omega_{[0,r]},$$

$$(3)$$

$$\frac{\partial u}{\partial n} + s\ (r,y)\ u\ =\ 0\ \text{on}\ H_{[0,r]}\ \times\ \partial G$$

and

$$M_1 v - \sum_{i,j=1}^{n} D_j(a_{ij}\,D_i v)\ +\ qv\ =\ 0\ \text{in}\ \Omega_{[0,r]},$$

(4)

$$\frac{\partial v}{\partial n} + t(r,y)\ v\ =\ 0\ \text{on}\ H_{[0,r]}\ \times\ \partial G$$

where s and t are functions of class $C^1(H_{[0,r]} \times \partial G)$ and $\dfrac{\partial}{\partial n}$ is the

transverse derivative defined by

$$\frac{\partial}{\partial n} = \sum_{i,j=1}^{n} a_{ij}(r,y)\ \upsilon_j\ D_i$$

$(\upsilon_1,...,\upsilon_n)$ being the outward unit normal vector on ∂G.

By setting $\varnothing - 1 + m = k$, the theorems 1,2, corollary 1, and theorem 3, given in [10] by Young for singular hyperbolic equations, can be generalized to the singular ultrahyperbolic equations given above, in the following theorems. Since the proofs are similar to those given in [10], they will be omitted.

THEOREM 1. Let $\varnothing > 1-m$, and assume that

$$p \leq q,\ s \leq t \qquad \text{in}\ G \tag{5}$$

If there exists a solution u (r,y) of the problem (3) which is positive in $\Omega_{(0,r)}$ such that $u(\Gamma,y) = 0$, then every solution v(r,y) of the problem (4) has a zero in $\Omega_{[0,r]}$

THEOREM 2. Let $\varnothing \leq 1-m$ and suppose that the condition (5) is satisfied. If u(r,y) is a solution of the adjoint of Lu = 0, that is

$$L^*u = u_{rr} - \left(\frac{k}{r}\,u\right)_r - \sum_{i,j=1}^{n} D_j(a_{ij}\,D_i u) + pu = 0\ \text{in}\ \Omega_{[0,r]}$$

which is positive in $\Omega_{(0,r)}$ and satisfies

$$u(o,y) = u(\Gamma,y) = 0 \text{ for } y \in G,$$

$$\frac{\partial u}{\partial n} + s(r,y) \, u = 0 \text{ on } H_{[o,r]} \times \partial G,$$

then every solution $v(r,y)$ of the problem (4) has a zero in $\Omega_{[o,r]}$.

If we consider the problems (3) and (4) in the domin $\Omega_{[\varepsilon,r]}$ where $\varepsilon > 0$, then a combination of the conditions in Theorems 1 and 2 leads to the following result which is valid for any value of the parameter $\varnothing$.

COROLLARY 1. If there exists a solution of $Lu = 0$ in $\Omega_{[\varepsilon,r]}$ which is positive in $\Omega_{(\varepsilon,r)}$ such that

$$u(\varepsilon,y) = u(\Gamma,y) = 0 \text{ in } G$$

$$\frac{\partial u}{\partial n} + s \, (r,y) \, u = 0 \text{ on } H_{[\varepsilon,r]} \times \partial G,$$

then every solution $v(r,y)$ of $Mv = 0$ in $\Omega_{[\varepsilon,r]}$ satisfying the condition

$$\frac{\partial v}{\partial n} + t(r,y) \, v = 0 \text{ on } H_{[\varepsilon,r]} \times \partial G$$

has a zero in $\Omega_{[\varepsilon,r]}$

The results given above, for the case $a_{ij} = b_{ij}$, can be extended to the cese when $a_{ij} \leq b_{ij}$, i,j $= 1,...,$n, provided that the coefficients a_{ij}, p, s and b_{ij}, q, t are all independent of the variables $x_1,...,x_m$, so that L and M are both seperable into their variables. Theorem 3 below includes this case.

THEOREM 3. Let $u(r,y)$ be a nontrivial solution of the problem

$$L_1 u - \sum_{i,j=1}^{n} D_j [a_{ij}(y) \, D_i u] + p(x)u = 0 \text{ in } \Omega_{[o,r]}$$

$$\frac{\partial u}{\partial n_a} + s(y) \, u = 0 \text{ on } H_{[o,r]} \times \partial G$$

such that $u(\Gamma,y) = 0$ for $\varnothing > 1-m$ or $u(0,y) = u(\Gamma,y) = 0$ for $\varnothing \leq 1-m$, where $p \geq 0$, $s \geq 0$ and p and s are not both identically zero. If

$$p(y) \leq q(y), \; s(y) \leq t(y), \; (a_{ij}(y)) \leq (b_{ij}(y)),$$

where at least one strict inequality holds throughout G, then every solution $v(r,y)$ of

$$M_1v - \sum_{i,j=1}^{n} D_j[b_{ij}(y) \ D_i v] + q(y) \ v = 0 \ \text{in} \ \Omega_{[0,r]}$$

$$\frac{\partial v}{\partial n_b} + t(y) \ v = 0 \ \text{on} \ H_{[0,r]} \times \partial G$$

where

$$\frac{\partial}{\partial n_a} = \sum_{i,j} a_{ij} \ \upsilon_j \ D_i, \quad \frac{\partial}{\partial n_b} = \sum_{i,j} b_{ij} \ \upsilon_j \ D_i,$$

has a zero in $\Omega_{(0,r)}$.

3. Oscillation Theorems.

We shall denote by Ω_δ the infinite domain

$$\Omega_\delta = \{(x,y): \delta \leq | \ x - \xi | \ < \infty, \ y \in G\}, \ \delta \geq 0,$$

where G is a bounded regular domain in R^n, and denote by Ω the domain Ω_0 for $\delta = 0$.

The elliptic operator P will be defined by

$$Pu \equiv - \sum_{i,j=1}^{n} D_j[a_{ij}(r,y) \ D_i u] + p(r,y)u$$

where x,y, r and D_i are defined as before in the introduction. We assume that the coefficients a_{ij} are in $C^1(\Omega)$ and that p(r,y) is continuous in Ω.

DEFINITION 1. A function $u: \Omega \rightarrow R$ is said to be oscillatory in Ω if u has a zero in Ω_δ for any $\delta > 0$.

DEFINITION 2. The differential equation Lu = 0 is said to be oscillatory in Ω if every solution u of Lu = 0, which vanishes on the lateral surface of Ω, is oscillatory in the interior of Ω.

In this section we shall give oscillation theorems for the singular ultrahyperbolic equation

$$Lu = L_1u + Pu = 0 \tag{6}$$

where the operator L_1 is defined as before:

$$L_1 u = \sum_{i=1}^{n} \left(u_{x_i x_i} + \frac{\alpha_i}{x_i - \xi_i} u_{x_i} \right) = u_{rr} + \frac{m-1+\sum_{i=1}^{m}\alpha_i}{r} u_r$$

We first establish an oscillation criteria in the case when the matrix (a_{ij}) of L is independent of x.

THEOREM 4. Let $p(y) \geq O$ and $s(y) \geq O$ in G and they are not both identicially zero. If $p(y) \leq q(r,y)$ in Ω and $s(y) \leq t(r,y)$ on $H_{[0,\infty)} \times \partial G$, then every solution $v(r,y)$ of

$$v_{rr} + \frac{m-1+\varnothing}{r} v_r - \sum_{i,j=1}^{n} D_j[a_{ij}(y) \, D_i v] + q(r,y) \, v = O$$

satisfying $\dfrac{\partial v}{\partial n} + t(r,y) \, v = O$ on $H_{[0,\infty)} \times \partial G$, has a zero in Ω_δ for every $\delta \geq O$. That is, the given equation is oscillatory in Ω.

Proof. We shall give the proof only for the case $\Sigma \alpha_i = \varnothing > 1-m$ since the proof for $\varnothing \leq 1-m$ is similar. Let $\Psi > O$ be the eigenfunction corresponding to the first eigenvalue λ_0 of the problem

$$- \sum_{i,j=1}^{n} D_j[a_{ij}(y) \, D_i w] + p(y)w = \lambda w \quad \text{in } G$$

$$\frac{\partial w}{\partial n} + s(y) \, w = O \qquad \text{on } \partial G.$$

Then for $k = m - 1 + \varnothing$

$$u(r,y) = r^{(1-k)/2} \, J_{(k-1)/2} (\sqrt{\lambda_0} \, r) \, \Psi(y)$$

is a nontrivial solution of problem (3) which vanishes at the sequence of points $r_1 < r_2 < \ldots < r_n < \ldots$, $r_n = z_n / \sqrt{\lambda_0}$ ($n \geq 1$), where z_n are the zeros of the Bessel function $J_{(k-1)/2}(z)$. The theorem then follows by applying Corollary 1 to each of the domains

$$\Omega_n = \{ (x,y): r_n \leq |x-\xi| \leq r_{n+1}, \ y \in G \}, \ n \geq 1.$$

Now we introduce the following notation:

$$\langle u(y), v(y) \rangle = \int_G u(y) \, v(y) \, dy.$$

THEOREM 5. Suppose that $v(r,y)$ is a function in $C^1(H_{(0,\infty)} \cap C(G))$ such that $v(r_0,y) = v(r_1,y) = O$ and that $h(y)$ is a positive function

in C(G). If $u(r,y)$ is a solution of (6) such that $< u(r,y),\ h(y) > \neq 0$ for $x \in H_{[r_0,r_1]}$ then

$$\int_{r_0}^{r_1} r^k \{ <v_r,h>^2 - <v,h> <Pu,h> <u,h>^{-1} \}\ dr\ >\ 0.$$

Proof. If $<u(r,y),h(y)>$ is not zero in $H_{[r_0,r_1]}$ then there exists a continuously differentiable function $w(r)$ such that

$$<v(r,y),\ b(y)>\ =\ w(r)\ <u(r,y),h(y)>.$$

The following identity can be established by differentiation:

$$r^k [w'(r) <u,,h>]^2 + \frac{d}{dr}\ [w(r) <v,h)> <r^k u_r, h>] \tag{7}$$

$$=\ r^k\ \{ <v_r,b>^2 - <v,h>^2 <Pu,h> <u,h>^{-1} \}$$

Integration of (7) in $H_{[r_0,r_1]}$ gives

$$\int_{r_0}^{r_1} r^k\ \{ <v_r,h>^2 - <v,h>^2 <Pu,h> <u,b>^{-1} \}\ dr$$

$$=\ \int_{r_0}^{r_1} r^k\ [w'(r) <u,b>]^2\ dr>\ 0.$$

The proof now follows.

We now define an admissible function $u(r,y)$ as a continuous function which is positive in Ω for large r and vanishes on the lateral boundary of Ω. Let $h(y)$ be a fixed positive function in $C(G)$ satisfying $<h,b> = 1$ and let $Q(r)$ be a continuous function satisfying the inequality

$$Q(r) \leq \frac{<Pu,h>}{<u,h>} \tag{8}$$

for large r and all admissible functions $u(r,y)$.

THEOREM 6. If the equation

$$\sum_{i=1}^{m} \left(Z_{x_i x_i} + \frac{\alpha_i}{x_i - \xi_i}\ Z_{x_i} \right) + Q(r)\ Z = 0 \tag{9}$$

is oscillatory in $H_{(0,\infty)}$, then the ultrahyperbolic equation (6) is oscillatory in Ω for $r = |x-\xi| > 0$.

Proof. Suppose (6) has a solution $u(r,y)$ which is positive for large r and vanishes on the lateral boundary of Ω; that is, $u(r,y)$ is an admissible function. Now since (9) is oscillatory there exists a solution $Z(r)$ of (9) such that $Z(r_0) = Z(r_1) = 0$ where r_0 and r_1 are arbitrarily large. We know that the equation (9) can be written as

$$Z_{rr} + \frac{k}{r} Z_r + Q(r) Z = 0, \ k = m - 1 + \sum_1^m \alpha_i \quad (9a)$$

If we let $v(r,y) = h(y) Z(r)$ then $v(r_0,y) = v(r_1,y) = 0$ and

$$\int_{r_0}^{r_1} r^k \langle v_r,h \rangle^2 - \langle v,h \rangle^2 \langle Pu,h \rangle \langle u,h \rangle^{-1} \} \ dr$$

$$\leq \int_{r_0}^{r_1} r^k \{Z'^2 - Q Z^2\} \ dr = \int_{r_0}^{r_1} (r^k ZZ')' \ dr = 0$$

This is in contradiction to Theorem 5.

DEFINITION 3. The elliptic equation

$$- \sum_{i,j=1}^n D_j [a_{ij}(y) D_i u] + p(y) u = 0 \quad (10)$$

is called disconjugate in G if the first boundary value problem for (10) is uniquely solvable for every smooth subdomain $G' \subset G$.

The above condition is clearly equivalent to the positivity of the first eigenvalue of

$$- \sum_{i,j=1}^n D_j [a_{ij}(y) D_i u] + p(y) u = \lambda u \text{ in } G$$

$$u = 0 \text{ on } \partial G.$$

THEOREM 7. The ultrahyperbolic equation

$$\sum_{i=1}^m \left(u_{x_i x_i} + \frac{\alpha_i}{x_i - \xi_i} u_{x_i} \right) - \sum_{i,j=1}^n D_j [a_{ij}(y) D_i u] + p(y) u = 0 \quad (11)$$

is oscillatory in Ω if and only if the uniformly elliptic equation

$$- \sum_{i,j=1}^n D_j [a_{ij}(y) D_i u] + p(y) u = 0 \quad (12)$$

is disconjugate in G.

Proof. Assume that the elliptic equation (12) is disconjugate in G, and let h(y) be a positive eigenfunction corresponding to the smallest positive eigenvalue λ_0 of

$$- \sum_{i,j=1}^{n} D_j [a_{ij}(y) \ D_j u] + p(y) \ u = \lambda u \ \text{in} \ G \qquad (13)$$

$$u = 0 \ \text{on} \ \partial G.$$

Then if u(r,y) is an admissible function we have

$$\lambda_o \leq \frac{\langle Pu,h \rangle}{\langle u,h \rangle}$$

It follows from Theorem 6 that equation (11) is oscillatory.

If equation (12) is not disconjugate in G, let h(y) be a positive eigenfunction corresponding to the smallest eigenvalue λ_0 of (13), which by assumption will be negative. Then

$$u(r,y) = r^{(1-k)/2} \ I_{(k-1)/2}(\sqrt{-\lambda_0} r) \ h(y)$$

is a solution of (11) which is not oscillatory. Here the function I_k is a modified Bessel function of the first kind of order k.

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ÖZET

Bu çalışmada singüler ultrahiperbolik kısmi türevli denklemlerin bir sınıfı için $H \times G \subset E^{m+n}$ bölgesinde karşılaştırma ve salınım teoremleri verilmiştir. E^m de, ξ eşmerkezli iki küre tarafından sınırlanan H bölgesinin noktaları $|x-\xi| = r$ yarıçap değişkeni cinsinden ifade edilerek, gözönüne alınan denklemin singüler hiperbolik denklemlere bir dönüştürmesi yapıldıktan sonra, sınırlı bir bölgede karşılaştırma teoremleri ifade edilmiş ve sonsuz bir bölgede salınım teoremleri ispatlanmıştır.