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On Equalizers And Coequalizers In Comma Categories

by

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On Equalizers And Coequalizers In Comma Categories

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Maclane [3] has discussed equalizers and coequalizers in comma categories ($\mathcal{C} \downarrow A$) and ($T \downarrow A$); we study their creation, reflection and preservation by the projection functors in the comma categories ($T \downarrow S$) and ($\mathrm{Cat} \cdot\!\downarrow\!\cdot \mathcal{C}$).

1. PRELIMINARIES

Definition 1.1.: Given categories and functors

$$\mathcal{B} \xrightarrow{\ T\ } \mathcal{C} \xleftarrow{\ S\ } \mathcal{D}$$

the (general) comma category ($T \downarrow S$) has as objects all triples (B,D,f) with $B \in |\mathcal{B}|$, $D \in |\mathcal{D}|$ and $f\colon TB \longrightarrow SD$; and as morphisms $(B,D,f) \longrightarrow (B',D',f')$ all pairs (u,v) of morphisms such that $f' o\, T u = Sv\, o\, f$. Diagramically:

$$\text{Objects}(B\,D, f) : \begin{array}{c} TB \\ \downarrow f \\ SD \end{array} ; \quad \text{morphisms}(u,v) : \begin{array}{ccc} TB & \xrightarrow{T(u)} & TB' \\ \downarrow f & & \downarrow f' \\ SD & \xrightarrow[S(v)]{} & SD' \end{array}$$

The composition $(u',v')o(u,v)$ is $(u'ou, v'ov)$, when defined.

Definition 1.2: Let $\mathcal{C}$ be an arbitrary category and assume that Cat denotes the category of small categories; then the category

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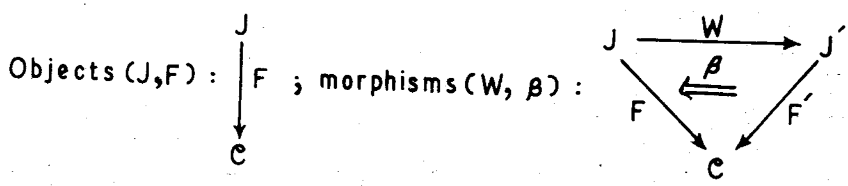
(Cat $\cdot\downarrow\cdot\mathcal{C}$) known as super–comma category in the sense of Maclane [3], has objects all pairs (J,F) where J is a small category and $F: J \longrightarrow \mathcal{C}$ a functor; and morphisms $(W, \beta): (J,F) \longrightarrow (J',F')$ are those pairs consisting of a functor $W: J \longrightarrow J'$ and a natural transformation $\beta: F'W \longrightarrow F$. Diagramically we have



The composition (W', β') o (W, β) is given by $(W'W, \beta\ \beta' W)$.

The category is also known as 'large diagram category' in the sense of Pareigis [4]. In fact, the concept of this category was introduced by Eilenberg and Maclane [1] pp. 277–280 and later on, was generalized by D.M. Kan [2].

2. COMMA CATEGORY (T $\downarrow$ S)

Let us define a functor $Q: (T \downarrow S) \longrightarrow \mathcal{B}\times\mathcal{D}$ as: Q assigns to each object (B,D,f) in (T $\downarrow$ S) the object (B,D) in $\mathcal{B}\times\mathcal{D}$ and to each morphism $(u,v): (B,D,f) \longrightarrow (B',D',f')$, the morphism $(u,v): (B,D) \longrightarrow (B', D')$.

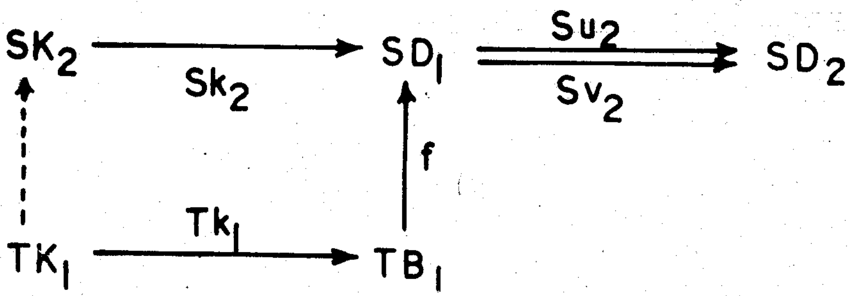
We now prove that

Theorem 2.1: $Q: (T \downarrow S) \longrightarrow \mathcal{B}\times\mathcal{D}$ creates equalizes if $\mathcal{D}$ is a category with equalizers and S preserves equalizers.

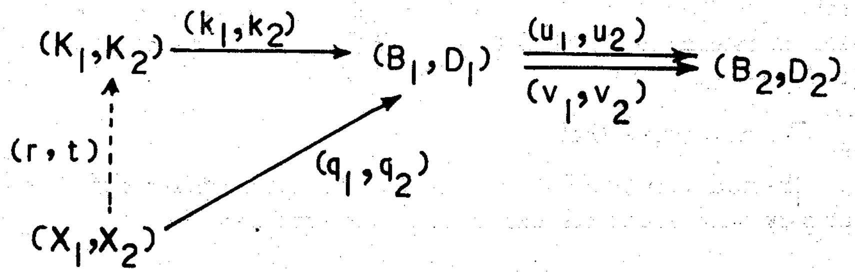
Proof: Let $(B_1,D_1,f) \underset{(v_1,v_2)}{\overset{(u_1,u_2)}{\rightrightarrows}} (B_2,D_2,g)$ be two morphisms in

(T $\downarrow$ S) and let (K_1,K_2) with (k_1,k_2) be the equalizer of (B_1,D_1)

$\underset{(v_1,v_2)}{\overset{(u_1,u_2)}{\rightrightarrows}}$ (B_2,D_2) in $\mathcal{B}\times\mathcal{D}$. If we consider



then, as S preserves equalizes $(SK_2;\ Sk_2)$ is the equalizer of Su_2, Sv_2 and moreover $Su_2f\ T\ k_1 = Sv_2f\ T\ k_1$; therefore by definition of equalizer, there exists a unique morphism, say, $k\colon TK_1 \longrightarrow SK_2$ making the above diagram commutative. It yields that (K_1,K_2,k) is an object of $(T \downarrow S)$ and (k_1,k_2) with $(u_1,u_2)(k_1,k_2) = (v_1,v_2)(k_1,k_2)$, is a morphism in $(T \downarrow S)$. We claim that (K_1,K_2,k) with (k_1,k_2) is the equalizer of (B_1,D_1,f) $\underset{(v_1,v_2)}{\overset{(u_1,u_2)}{\rightrightarrows}}$ (B_2,D_2,g). Consider (X_1,X_2,h) and $(q_1,q_2)\colon (X_1,X_2,h) \longrightarrow (B_1,D_1,f)$ with $(u_1,u_2)(q_1,q_2) = (v_1,v_2)(q_1,q_2)$. Then if we consider



there is a unique morphism, say, $(r,t)\colon (X_1,X_2) \longrightarrow (K_1,K_2)$ making the above diagram commutative. Moreover, we have

$$Sk_2kTr = fTk_1Tr = f\ Tq_i = Sq_2h = Sk_2Sth$$

thus if follows that $kTr = Sth$ and therefore $(r,t) \in (T \downarrow S)$. Hence Q creates equalizers.

Dually, we can prove the following:

Theorem 2.2: $Q: (T \downarrow S) \longrightarrow \mathcal{B}\times\mathcal{D}$ creates coequalizers if $\mathcal{B}$ is a category with coequalizers and T preserves coequalizers.

The following corollaries can be deduced from these theorems.

Corollary 2.3: $Q: (T \downarrow S) \longrightarrow \mathcal{B}\times\mathcal{D}$ reflects equalizers and coequalizers.

Corollary 2.4: $Q: (T \downarrow S) \longrightarrow \mathcal{B}\times\mathcal{D}$ preserves equalizers if Q creates equalizers and S preserves those equalizers; Dually, for coequalizers.

3. COMMA CATEGORY (Cat $\cdot\downarrow\cdot\mathcal{C}$)

Define a functor $Q: (\text{Cat} \cdot\downarrow\cdot\mathcal{C}) \longrightarrow \text{Cat}$ such that Q sends each object (J,F) of (Cat $\cdot\downarrow\cdot\mathcal{C}$) to J of Cat and each morphism (W, β): $(J, F) \longrightarrow (J',F')$ to $W: J \longrightarrow J'$.

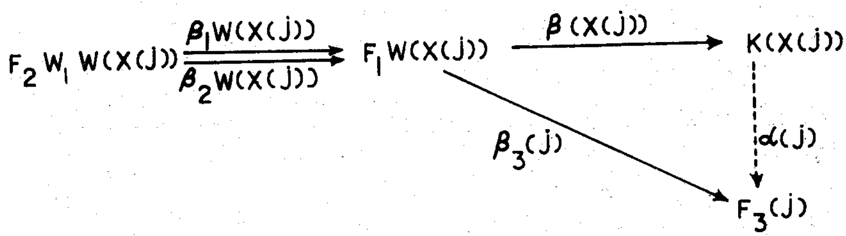
Now we show that.

Theorem 3.1: $Q: (\text{Cat} \cdot\downarrow\cdot\mathcal{C}) \longrightarrow \text{Cat}$ creates equalizers if $\mathcal{C}$ is a category with coequalizers.

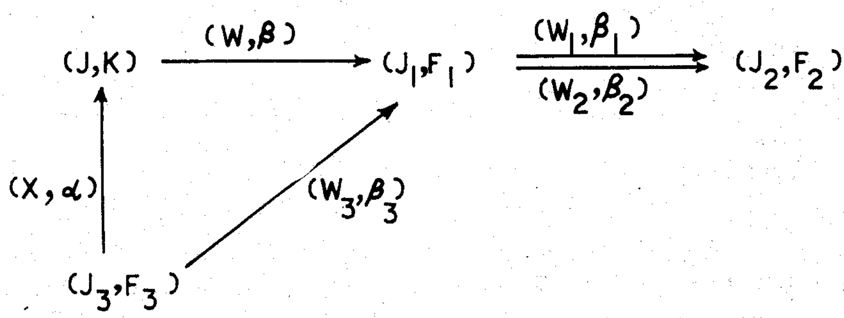
Proof: Let $(J_1,F_1) \underset{(W_2,\beta_2)}{\overset{(W_1,\beta_1)}{\rightrightarrows}} (J_2,F_2)$ be two morphisms in

(Cat $\cdot\downarrow\cdot\mathcal{C}$) and (J;W)be the equalizer of $Q(W_1,\beta_1)$, $Q(W_2,\beta_2)$ i.e. of W_1 and W_2 in Cat; then $W_1W = W_2W$. Since $\beta_1: F_2W_1 \longrightarrow F_1$ and $\beta_2: F_2W_2 \longrightarrow F_1$ are natural transformations, it follows that so are $\beta_1W: F_2W_1W \longrightarrow F_1W$, $\beta_2W: F_2W_2W \longrightarrow F_1W$. As $\mathcal{C}$ is a category with coequalizers, $[J, \mathcal{C}]$ (the category of functors from J to $\mathcal{C}$) is also a category with coequalizers. Let $(K; \beta)$ be the coequalizer of β_1W, β_2W, i.e. $\beta\ \beta_1W = \beta\ \beta_2W$; so that (J,K) is an object in (Cat $\cdot\downarrow\cdot\mathcal{C}$) and $(W, \beta): (J,K) \longrightarrow (J_1,F_1)$ is a morphism in (Cat $\cdot\downarrow\cdot\mathcal{C}$) with $(W_1,\beta_1) \circ (W, \beta) = (W_2, \beta_2) \circ (W, \beta)$. Consider (J_3,F_3) in (Cat $\cdot\downarrow\cdot\mathcal{C}$) and a morphism (W_3, β_3): $(J_3,F_3) \longrightarrow (J_1,F_1)$ such that $(W_1,\beta_1) \circ (W_3,\beta_3) = (W_2,\beta_2) \circ (W_3,\beta_3)$; i.e.

$(W_1 x W_3, \beta_3 x \beta_1 x W_3) = (W_2 W_3, \beta_3 \beta_2 W_3)$. Hence by definition of equalizer there is a unique morphism (functor) $X: J_3 \longrightarrow J$ in Cat such that $WX = W_3$. Replace W_3 by WX and for any $j \in J_3$, consider the following diagram



Then there exists a unique natural transformation $\alpha: KX \longrightarrow F_3$ such that $\alpha \beta X = \beta_3$. We have thus obtained a unique morphism $(X, \alpha): (J_3, F_3) \longrightarrow (J,K)$ such that $(W, \beta) \circ (X, \alpha) = (W_3, \beta_3)$



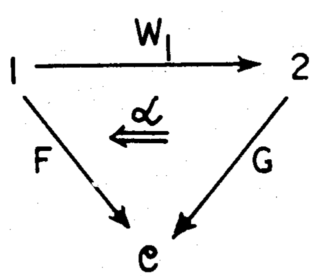
The following can be proved as Corollaries to the above theorem.

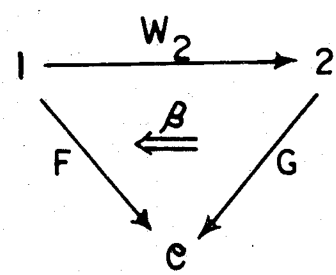
Corollary 3.2: $Q: (\text{Cat} \cdot\downarrow\cdot \mathcal{C}) \longrightarrow \text{Cat}$ reflects equalizers if $\mathcal{C}$ is a category with coequalizers.

Corollary 3.3: $Q: (\text{Cat} \cdot\downarrow\cdot \mathcal{C}) \longrightarrow \text{Cat}$ preserves equalizers if $\mathcal{C}$ is a category with coequalizers.

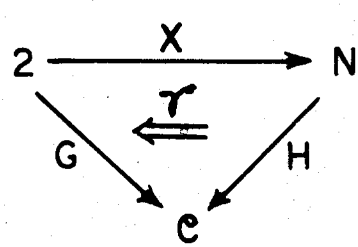
However, the corresponding results for coequalizers do not hold. We give an example in support of our statement.

Example 3.4: Suppose that $\mathcal{C}$ is the category $C_1 \overset{p}{\underset{q}{\rightrightarrows}} C_2$. Let 1 be the unit category, and 2 the category $A \xrightarrow{f} B$. Let $W_1, W_2 \colon 1 \longrightarrow 2$ be the functors sending 1 to A and B respectively; their coequalizer in Cat is $X \colon 2 \longrightarrow N$, where N has one object * and its morphisms are the free abelian monoid on t, say, and $X(f) = t$. Let $F \colon 1 \longrightarrow \mathcal{C}$ send 1 to C_2; and $G \colon 2 \longrightarrow \mathcal{C}$ be the constant functor at C_1; and let natural transformations α and β





be given by $\alpha 1 = p$, $\beta 1 = q$. If Q created coequalizers, the coequalizer of (W_1, α) and (W_2, β) would have the form



This is impossible, for $\alpha . \gamma W_1$ and $\beta . \gamma W_2$ are different.

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