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Algebraic Structure of the Restricted Analytic Sheaves and Restricted Ideal Sheaves

by

Cengiz ULUÇAY

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

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Algebraic Structure of the Restricted Analytic Sheaves and Restricted Ideal Sheaves

Cengiz ULUÇAY

Dept. of Mathematics, Faculty of Science, Institute of Science, Ankara University, Ankara, Turkey.

SUMMARY

In this paper algebraic structure of the restricted analytic sheaves and restricted Ideal sheaves recently introduced by the Author is studied.

1. **Restricted Analytic Sheaves and Ideal Sheaves.** In the papers [1,2,3] we have studied the restricted analytic sheaves and subsheaves (also called restricted Ideal sheaves). There are regular covering spaces of a given complex analytic manifold X of dimension n with fundamental group $F \neq 1$. The fundamental groups of these regular spaces project onto and are isomorphic to a normal subgroup D of F such that F/D is Abelian. Conversely, to a normal subgroup D of F such that F/D is Abelian corresponds a restricted anlytic sheaf or subsheaf [3]. As a consequence the group T of cover transformations of these regular spaces are on the one hand isomorphic to the quotient group F/D and on the other hand to the additive group of sections over X. From these considerations the fundamental theorem [3] and the earlier results [1, 2] follow easily. Namely.

$$\Gamma(X,\mathcal{A}) \cong A(X) \cong F/[F,F] \cong T$$

and

$$\Gamma(X,\mathcal{A}') \cong A'(X) \cong F/D \cong T'.$$

Here the $\mathcal{A}$'s are the restricted analytic sheaves, the Γ's are the group of sections over X, A(X) is the totality of holomorphic functions on X, A′(X) is an ideal of A(X) and the T's are the groups of corresponding cover transformations, respectively.

It is now the purpose of this paper to show in some details the algebraic foundation of these results.

2. **Algebraic Notions.** [4,5].

Coset. If H is a subgroup, and x an element of a group G, then the product Hx (H+x in additive notation) of the set H with the set consisting of the single element x is the right coset of H in G and containing x. The left coset xH is defined similarly.

If $H \subset G$ is a subgroup, a,b $\in$ G are called right equivalent modulo H if $ab^{-1} \in H$. The right equivalenrce is an equivalence relation on G. Two elements are in the same right coset if and only if they are right equivalent modulo H. Since each element $x \in G$ is in a right coset of H, i.e., Hx we have:

If $H \subset G$ is a subgroup then G is the union of the right cosets, two right cosets of H in G are either disjoint or identical.

The element $x = 1x$ belongs to the coset Hx and is called the representative of the coset. Any element $u \in Hx$ may be taken as the representative, since $Hu = Hx$. $H1 = H$ is one of its own cosets. Here the identity 1 is the representative of H. We may therefore write that G is the disjoint union of $H = H1$ and the Hx's for all $x \in G$, such that $x \notin H$. Here, and in the sequel, $x \notin H$ will mean any one of $x_1, x_2, \ldots$ in the disjoint union $G = H1 \cup Hx_1 \cup Hx_2 \cup \ldots$.

Normal Subgroup.

Definition 2.1. If $H \subset G$ is a subgroup, then H is normal in G if $xH = Hx$ for all $x \in G$. In the sequel a normal subgroup will be denoted by N.

Definition 2.2. Let N be a normal subgroup of G. Then the set of all cosets is called the factor or the quotient group of G by the normal subgroup N and is denoted by G/N.

Proposition 2.1. Let N be a normal subgroup of G, and $x \in G$, $x \notin N$. Then the representative 1 of N1 and the representatives x of distinct cosets Nx of G/N form a group isomorphic to G/N.

Proof. Each representative $x \in Nx$ and 1 may be viewed as a transform $x: N \to Nx$, $1: N \to N1$. There is a one to one correspondence between the elements 1 and $x \in Nx$ on the one hand and the cosets $N1$ and Nx on the other hand, the identity 1 corresponding to $N1$. Moreover, to the product xy corresponds the product $NxNy$. For, $xy: N \to Nxy = NNxy = NxNy$ Hence there is an isomorphism between the elements 1 and $x \in Nx$ on the one hand and the cosets $N1$, Nx on the other hand respectively. Since the set of all distinct cosets is the factor group G/N, the proposition follows.

Proposition 2.2. Let $N \subset G$ be a normal subgroup, and $\Psi: G \to G^*$ a homomorphism of G onto G^* with kernel N. Then $G/N \cong G^*$.

Proof. Let $a^* \in G^*$ and A be the totality of all the elements of G which map onto a^* under Ψ, i.e., $\Psi(A) = a^*$. While $\Psi(N) = 1^*$ (the unit element of G^*) by hypothesis. If $a, a' \in A$ are two elements, then $\Psi(a'a^{-1}) = \Psi(a')\,\Psi(a^{-1}) = \Psi(a')\,(\Psi(a))^{-1} = a^*a^{*-1} = 1^*$.

Hence $a'a^{-1} \in N$ and so a, a' belong to the same coset of N, say Nx. Thus $A \subset Nx$. Conversely, if b belongs to the same coset Nx as a, i.e., $ba^{-1} \in N$, then

$$1^* = \Psi(ba^{-1}) = \Psi(b)\,\Psi(a^{-1}) = \Psi(b)\ \ a^{*-1}.$$

Hence $\Psi(b) = a^*$ and so $b \in A$. Therefore $Nx \subset A$. In conclusion $Nx = A$. Namely, there is complete identity between A and the coset Nx containing $a \in A$. Furthermore, Ψ yields a one to one correspondence f between the cosets Nx (or x) and the elements of G^*. Namely, $f(1) = 1^*$ or $f(N1) = 1^*$, $f(x) = a^*$ or $f(Nx) = a^*$. Finally ,f is an isomorphism. For, let $y \in G$ and $y \notin N$, i.e., Ny. We have $f(Nx) = a^*$, $f(Ny) = b^*$. But $Nxy = NxNy$ and

$$f(Nxy) = f(NxNy) = f(Nx)\,f(Ny) = a^*b^*$$

or, in view of proposition 2.1, also

$$f(xy) = a^*b^*.$$

Hence, $G^* \cong G/N$.

Definition 2.3. Let G be a group and $S \subset G$ a subgroup. The conjugate of S by $x \in G$ is the set $x^{-1}Sx$ of all $x^{-1}sx$, $s \in S$. If $x^{-1}Sx = S$ for all $x \in G$, then S is normal and will be denoted as usual by N. For fixed $x \in G$ the set $x^{-1} Nx$ of all conjugates $x^{-1} nx$, $n \in N$ is the conjugate class or the transform of N by x. For a normal N we have

$$x^{-1} Nx = N, \forall x \in G.$$

This means that N remains invarient under all conjugations $x^{-1} Gx$ (inner automorphisms) of G.

Hence

$$Nx = xN, \forall x \in G.$$

Two elements $x, y \in G$ are said to be conjugate if there is a $z \in G$ so that $x = z^{-1} yz$. This notion aso leads to an equivalence relation on G, and hence that G is the disjoint union of the conjugate classes of N just as it is the disjoint nion of the cosets of N.

Proposition 2.3. The set of all conjugate classes of N forms a group isomorphic to the factor group G/N.

Proof. If for fixed $x,y \in G$, $x^{-1} Nx = N$, and also $y^{-1} Ny = N$, then $(xy)^{-1} N(xy) = N$. But the above relations just define cosets and product of cosets respectively. Hence the set of conjugate classes of N is isomorphic to the set of cosets of N and the proposition follows. Here again one may distinguish the elements $x \in G$ not in N.

Commutator Subgroup.

Definition 2.4. Let G be a non commutative group. The commutator of G, denoted by $[G,G]$ is defined as the smallest subgroup which contains all elements of the form $aba^{-1} b^{-1}$ also denoted by $[a, b]$, $a, b \in G$. It is easy to see that $[G, G]$ is a normal subgroup and that $G/[G,G]$ is Abelian. Moreover it is the smallest subgroup whose quotient group is Abelian.

Theorem 2.1. Let G,G' be two noncommutative groups. Then $[G,G] \cong [G',G']$ if and only if $G \cong G'$.

Proof. It is sufficient to show that the isomotphism φ between the groups implies the isomorphism between the commutator subgroups.

Since $\alpha = aba^{-1} b^{-1}$ is also in G, then

$$\varphi(\alpha) = \varphi(aba^{-1}b^{-1}) = \varphi(a)\, \varphi(b)\, (\varphi(a))^{-1} (\varphi(b))^{-1}$$
$$= a'b'a'^{-1}\, b'^{-1} = \alpha' \in [G',G'].$$

3. Applications to restricted analytic sheave.

Suppose that the group G under investigation is the fundamental group F $\neq$ 1 of a complex analytic manifold X of dimension n. The elements of F are now the closed arcs γ that begin and end at a fixed point in X [1]. Here a normal subgroup of F is denoted by D and is such that F/D is Abelian. To a coset or conjugate class of the normal subgroup D is associated a section over X [1] of the restricted analytic sheaf or subsheaf whose fundamental group projects onto and is isomorphic to the normal subgroup D of F. The group of cover transformations is isomorphic to F/D which in turn is isomorphic to the group of the transforms of a section by the closed arcs not belonging to D. From these observations the main theorems proved in [1,2,3] follow at once. These theorems have been recollected at the beginning in the first section: Restricted analytic sheaves and Ideal sheaves. Finally, the counterpart of theorem 2.1 is just the second corollary of theorem 6.5 in [1].

The results of the papers [1,2,3] may be summarized as follows:

Theorem 3.1. Let X be a connected complex analytic manifold with fundamental group F $\neq$ 1. Let D denote any *normal* subgroup of F such that F/D is commutative. Then [F, F] is the smallest normal subgroup of that type. Each D determines a regular covering space of X with fundamental group D and which is a restricted ideal subsheaf of the restricted analytic sheaf $\mathcal{A}$ of the germs of the totality A (X) of holomorphic functions on X. In particular, [F,F] determines $\mathcal{A}$ which is maximal.

The proof of this theorem is virtually contained in the quoted papers and especially in the fundamental theorem [3]. On the other hand [F,F] $\subset$ D is clear. For otherwise, there is [a,b] $\notin$ D, a,b $\in$ F. The elements of F/D being the cosets of D, let A,B,A^{-1}, B^{-1} be that ones concontaining a,b,a^{-1}, b^{-1} respectively. But then [A,B] is a coset containing [a,b] and is different from D, and so A,B do not commute, contrary to the hypothesis. Finally, F/[F,F] is commutative. For if A,B are any two cosets of [F,F] then [A,B] is a coset containing a commutator [a, b], a $\in$ A, b $\in$ B and so [A,B] $\equiv$ [F,F].. Hence A,B commute.

Summarizing we obtain the following

Criterion: F/D is commutative (or a regular covering space of X determined by a normal subgroup D is r-ideal subsheaf of A) if and only if D contains [F,F].

We finally give examples of *normal* subgroups of F that do not contain [F,F]. Indeed, if N is any normal subgroup of F, so is [N,N]. But clearly, [N,N] does not contain [F,F]. Hence. not every regular covering space of X is a r-ideal subsheaf.

ÖZET

Bu makalede, son zamanlarda Müellif tarafından geliştirilen Tahditli Analitik Demetlerin ve Tahditli İdeal Demetlerin Cebirsel yapıları incelenmektedir.

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