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A Note On L^1-Convergence Of Fourier Series With δ-Quasi-Monotone Coefficients

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A Note On L^1-Convergence Of Fourier Series With δ-Quasi-Monotone Coefficients

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ABSTRACT

For the class of Fourier series with δ-quasimonotone coefficients, it is proved that $\| S_n - \sigma_n \| = o(1), n \to \infty$, if and only if $a_n \log n = o(1), n \to \infty$. This generalizes the theorem of Garrett, Rees and Stanojevic [3], and Telyakovskii and Fomine [6] for quasi-monotone, and monotone coefficients respectively.

1. A sequence $\{a_n\}$ of positive numbers is said to be quasi-monotone if $\Delta a_n \geq -\alpha \frac{a_n}{n}$ for some positive α, where $\Delta a_n = a_n - a_{n+1}$. It is obvious that every null monotonic decreasing sequence is quasi-monotone. The sequence $\{a_n\}$ is said to be δ-quasi-monotone if $a_n \to o$, $a_n > o$ ultimately and $\Delta a_n \geq -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers. Clearly a null quasi-monotone sequence is δ-quasi-monotone with $\delta_n = \alpha \frac{a_n}{n}$.

2. The problem of L^1-convergence of Fourier cosine series

$$f(x) = \tfrac{1}{2} a_o + \sum_{n=1}^{\infty} a_n \cos nx$$

has been settled for various special class of coefficients, (See e.g. Young [7], Kolmogorov [4], Fomine [1], Garrett and Stanojevic [2], Telyakovskii and Fomine [6], etc).

Recently, Garrett, Rees and Stanojevic [3] proved the following theorem which is too a generalization of a result of Telyakovskii and Fomine ([6], Theorem 1).

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THEOREM A. Let $f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ be a Fourier series with quasi-monotone coefficients. Then

$$||S_n - \sigma_n|| = o(1), n \to \infty,$$

if and only if

$$(a_n + b_n) \log n = o(1), n \to \infty.$$

Where σ_n is the Fejĕr sum, $||\ .\ ||$ is the L^1-norm and

$$S_n (x) = \frac{1}{2} a_o + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

We propose to generalize this result by replacing the quasi-monotonicity of the coefficients by its δ-quasi-monotonicity.

3. We prove the following theorem

THEOREM. *Let* $f(x) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ *be a Fourier series with δ-quasi-monotone coefficients with* $\Sigma\, n\, \delta_n < \infty$. *Then*

$$||S_n - \sigma_n|| = o(1), n \to \infty,$$

if and only if

$$(a_n + b_n) \log n = o(1), n \to \infty,$$

where S_n *and* σ_n *are the same as in Theorem A.*

4. For the proof of the theorem we require the following lemmas.

LEMMA 1. [5] *If the sequence* $\{a_n\}$ *is δ-quasi-monotone and* $\Sigma\, a_n \Delta\Phi_n$ *converges, then* $a_n \Phi_n \to o$, *as* $n \to \infty$, Φ_n being a positive monotone increasing sequence.

LEMMA 2. *Let* $\{a_n\}$ *be a δ-quasi-monotone sequence with* $\Sigma\, n\, \delta_n < \infty$. If $a_n \log n = o(1), n \to \infty$, then

$$\frac{1}{n} \sum_{k=1}^{n} k\, |\Delta a_k| \log k = o(1), \ n \to \infty.$$

Proof. From

$$\frac{1}{n}\sum_{k=1}^{n} a_k = \frac{1}{n}\sum_{k=1}^{n-1} \Delta (ka_k) \sum_{j=1}^{k} \frac{1}{j} + a_n \sum_{j=1}^{n} \frac{1}{j} ,$$

we obtain

$$\frac{1}{n}\sum_{k=1}^{n-1} k\,\Delta a_k \sum_{j=1}^{k} \frac{1}{j} = \frac{1}{n}\sum_{k=1}^{n} a_k - a_n \sum_{j=1}^{n} \frac{1}{j}$$

$$+ \frac{1}{n}\sum_{k=1}^{n-1} a_{k+1} \sum_{j=1}^{k} \frac{1}{j}$$

Since $\{a_n\}$ is δ-quasi-monotone, we have

$$|\Delta a_n| \leq \Delta a_n + 2\delta_n ,$$

where δ_n is a sequence of positive numbers. Hence,

$$\frac{1}{n}\sum_{k=1}^{n-1} k\,|\Delta a_k| \sum_{j=1}^{k} \frac{1}{j} \leq \frac{1}{n}\sum_{k=1}^{n-1} k\,(\Delta a_k + 2\delta_k)$$

$$\sum_{j=1}^{k} \frac{1}{j} = \frac{1}{n}\sum_{k=1}^{n-1} k\,\Delta a_k \sum_{j=1}^{k} \frac{1}{j}$$

$$+ \frac{2}{n}\sum_{k=1}^{n-1} k\,\delta_k \sum_{j=1}^{k} \frac{1}{j} \leq$$

$$\frac{1}{n}\sum_{k=1}^{n} a_k - a_n \sum_{j=1}^{n} \frac{1}{j} + \frac{1}{n}\sum_{k=1}^{n-1} a_{k+1} \sum_{j=1}^{k} \frac{1}{j} +$$

$$\frac{2}{n}\sum_{k=1}^{n-1} k\,\delta_k \sum_{j=1}^{k} \frac{1}{j}.$$

By hypotheses, each term on the right hand side in $o(1)$, $n \to \infty$. This completes the proof of Lemma 2.

5. *Proof of the theorem.* We shall carry out the proof for the cosine series only, the proof for the sine series being essentialy the same. Since σ_n is the Fejĕr sum and S_n is the n^{th} partial sum of the cosine series, following [3], we obtain,

$$||\, S_n - \sigma_n \,|| = \frac{1}{n+1} \,||\, \sum_{k=1}^{n} k\, a_k \cos kx \,||$$

$$\leq \frac{1}{n+1} \left|\left| \sum_{k=1}^{n-1} k \, \Delta a_k \, [D_k(x) - \tfrac{1}{2}] \right|\right|$$

$$+ \frac{1}{n+1} \left|\left| \sum_{k=1}^{n-1} a_{k+1} \, [D_k(x) - \tfrac{1}{2}] \right|\right|$$

$$+ a_n \left|\left| D_n(x) - \tfrac{1}{2} \right|\right|,$$

where $D_n(x)$ is the Dirichlet's Kernal. Since $|| D_n(x) - \frac{1}{2} || = O(\log n)$ for some $B > o$

$$B \, || S_n - \sigma_n || \leq \frac{1}{n+1} \sum_{k=1}^{n-1} k \, | \Delta a_k | \log k \, +$$

$$\frac{1}{n+1} \sum_{k=1}^{n-1} a_{k+1} \log k + a_n \log n.$$

From Lemma 2, it follows that

$$|| S_n - \sigma_n || = o(1), \; n \to \infty.$$

For the "only if" part, again following [3], we have

$$|| S_n - \sigma_n || + || \sigma_n - f || \geq || S_n - f ||$$

$$\geq C \sum_{k=1}^{n} \frac{a_{n+k}}{k}$$

$$\geq C \sum_{k=n+1}^{2n} \frac{a_k}{k},$$

where C is a positive constant, since $f \in L^1$, we have that $|| \sigma_n - f || = o(1), n \to \infty$. Assume $||S_n - \sigma_n || = o(1), n \to \infty$. Then

$$\sum_{k=n+1}^{2n} \frac{a_k}{k} = o(1), \; n \to \infty,$$

which by Cauchy convergence of infinite series, gives

$$\sum_{k=1}^{\infty} \frac{a_k}{k} < \infty.$$

Then, by the fact that $\{a_n\}$ is δ-quasi-monotone such that $\Sigma \, \delta_n \log n < \infty$, we get

$$a_n \log n = o(1), \; n \to \infty,$$

by an appeal to lemma 1, taking $\Phi_n = \log n$.

This completes the proof of the theorem.

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