COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1 : Mathématiques

TOME 31 ANNÉE 1982

A Note On A Theorem Of Izumi

by

Huzoor H. KHAN

14

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences
de l'Université d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les diciplines scientifiques représentées à la Faculté des Sciences de l'Université d'Ankara.

La Revue, jusqu'à 1975 à l'exception des tomes I, II, III etait composée de trois séries

Série A : Mathématiques, Physique et Astronomie,
Série B : Chimie,
Série C : Sciences Naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A_1 : Mathématiques,
Série A_2 : Physique,
Série A_3 : Astronomie,
Série B : Chimie,
Série C_1 : Géologie,
Série C_2 : Botanique,
Série C_3 : Zoologie.

En principe, la Revue est réservée aux mémoires originaux des membres de la Faculté des Sciences de l'Universiteé d'Ankara. Elle accepte cependant, dans la mesure de la place disponible les communications des auteurs étrangers. Les langues Allemande, Anglaise et Française seront acceptées indifféremment. Tout article doit être accompagné d'un resumé.

Les articles soumis pour publications doivent être remis en trois exemplaires dactylographiés et ne pas dépasser 25 pages des Communications, les dessins et figures portés sur les feuilles séparées devant pouvoir être reproduits sans modifications.

Les auteurs reçoivent 25 extraits sans couverture.

l'Adresse : Dergi Yayın Sekreteri
Ankara Üniversitesi
Fen Fakültesi
Beşevler–Ankara

A Note On A Theorem Of Izumi

Huzoor H. KHAN

(Received June 9, 1982; accepted December 31, 1982)

ABSTRACT

If $f(x) \in \text{Lip}\,\alpha$ $(0 < \alpha \leq 1)$ and $S_n(x)$, the n-th partial sum of its Fourier series, then

$$f(x) - S_n(x) = 0\,(1/n^{\alpha})$$

is not true in general but if $f(x) \in \text{Lip}\,(\alpha, p)$ then

$$f(x) - S_n(x) = 0\left(\frac{1}{n^{\alpha - 1/p}}\right).$$

Defining a new general class $W'(L^p, \psi(h))$, we examined that

$$f(x) - S_n(x) = O\,(\psi(1/n)\,n^{1/p}),$$

where $\psi(h)$ is a positive increasing function.

It may be remarked that the class $W'(L^p, \psi(h))$ is a more general class than Lip α, Lip (α, p), Lip $(\psi(h), p)$.

1. Introduction And Results

Let f be periodic with period 2π, and integrable in the sense of Lebesgue. The Fourier series associated with f at the point x, is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n x + b_n \sin n x) \equiv \sum_{\nu=1}^{\infty} A_\nu(x) \tag{1.1}$$

$$\text{Let } S_n(x) = \tfrac{1}{2} a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{1.2}$$

denote the n-th partial sum of the Fourier series (1.1) A function f (x) is said to belong to the class Lip (α, p), if

$$\left\{ \int_0^{2\pi} | f(x+h) - f(x) |^p \, dx \right\}^{1/p} = 0 (h^\alpha), \ 0 < \alpha \leq 1.$$

We define a new class named weighted (L^p, ψ (h)) (written W′ (L^p, ψ (h)) class and say f (x) $\in$ W′ (L^p, ψ (h)) if

$$\left\{ \int_0^{2\pi} | f(x+h) - f(x) |^p \operatorname{Sin}^{\beta p} \frac{x}{2} \, dx \right\}^{1/p} = 0 (\psi (h) h^\beta),$$

for $p > 1$ and $\beta \geq 0$, where ψ (h) is a positive increasing function.

The class W′ (L^p, ψ (h)) is a more general class than Lip (α, p) and reduces to class Lip (α, p), when $\psi (h) = h^\alpha$ and $\beta = 0$.

We write $\Phi_x (u) = \frac{1}{2} \{ f(x+u) + f(x-u) - 2 f(x) \}$. It can be easily proved that if f (x) $\in$ W′ (L^p, ψ (h)) then $\Phi_x (u) = 0 (\psi (u) u^\beta)$.

Izumi [1] has proved the following theorem:

Theorem I: If f (x) $\in$ Lip (α, p), where $0 < \alpha \leq 1, p > 1, \alpha p > 1$. then

$$f(x) - S_n(x) = 0 \left(\frac{1}{n^{\alpha - 1/p}} \right) \tag{1.3}$$

uniformly almost every where.

If f (x) $\in$ Lip α ($0 < \alpha \leq 1$) and S_n (x) is the n-th partial sum of its Fourier series, then

$$f(x) - S_n(x) = 0 \left(\frac{1}{n^\alpha} \right) \tag{1.4}$$

is not true in general, but if f (x) $\in$ Lip (α, p) then (1.3) holds uniformly almost every where. The purpose of this note is to extend Theorem I for the newly defined class W′ (L^p, ψ (h)).

It can be easily seen that the order arrived at is better than the order obtained by Izumi [1] and is free from any condition on β.

Our Theorem states as follows:

Theorem H: If f (x) $\in$ W′ (L^p, ψ (h)) class, such that

$$\left\{ \int_0^{\pi/n} \left(\frac{\psi (u)}{u^{\beta + 1 - \delta}} \right)^q du \right\}^{1/q} = 0 \left(\psi \left(\frac{1}{n} \right) n^{\beta - \delta + 1/p} \right) \tag{1.5}$$

where δ is an arbitary positive number such that $\delta \leq \frac{1}{p}$, then

$$f(x) - S_n(x) = 0 \left(\psi\left(\frac{1}{n}\right) n^{1/p}\right) \tag{1.6}$$

It may be remarked that Theorem H reduces to Theorem I if we put $\psi\left(\frac{1}{n}\right) = n^{-\alpha}$ in Theorem H.

2. Proof Of Theorem

We know that

$$f(x) - S_n(x) = \frac{1}{\pi} \int_0^{\pi} \Phi_x(u) \frac{\text{Sin } n u}{u} du$$

$$= \frac{1}{\pi} \left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right] \Phi_x(u) \frac{\text{Sin } n u}{u} du$$

$$= H_1 + H_2 \text{ (Say)}$$

In order to evaluate H_1, we proceed as follows:

$$| H_1 | = \frac{1}{\pi} \left| \int_0^{\pi/n} \Phi_x(u) \frac{\text{Sin } n u}{u} du \right|$$

$$| H_1 | \leq \int_0^{\pi/n} \frac{| \Phi_x(u) |}{u} du$$

$$= \int_0^{\pi/n} \left(\frac{u^{-\delta} \, | \Phi_x(u) | \, | \text{Sin}^{\beta} \frac{u}{2} |}{\psi(u)} \right) \left(\frac{\psi(u)}{u^{-\delta+1} \, | \text{Sin}^{\beta} \frac{u}{2} |} \right) du$$

Applying Hölder's inequality and the fact that $\Phi_x(u) = 0 (\psi(u) \, u^{\beta})$, we have

$$| H_1 | \leq \left[\int_0^{\pi/n} \left\{ \frac{(u^{-\delta} \quad \psi (u) \ u^{\beta}) \mid \operatorname{Sin}^{\beta} \frac{u}{2} \mid}{\psi (u)} \right\}^p du \right]^{1/p}$$

$$\times \left[\int_0^{\pi/n} \left\{ \frac{\psi (u)}{u^{1-\delta} \mid \operatorname{Sin}^{\beta} \frac{u}{2} \mid} \right\}^q du \right]^{1/q}$$

$$\leq \left[\left\{ \int_0^{\pi/n} u^{\beta p - \delta p} \ du \right\}^{1/p} \times \left\{ \int_0^{\pi/n} \left(\frac{\psi (u)}{u^{1-\delta}} \right) \left(\frac{\pi}{u} \right)^{\beta q} du \right\}^{1/q} \right]$$

$$= 0 \ (n^{\delta - \beta - 1/p}). \ 0 \left[\int_0^{\pi/n} \left(\frac{\psi (u)}{u^{\beta + 1 - \delta}} \right)^q du \right]^{1/q}$$

$$= 0 \ (n^{\delta - \beta - 1/p}). \ 0 \left(\psi \ \left(\frac{1}{n} \right). \ n^{\beta - \delta + 1/p} \right) \quad [\text{by condition } (1.5)]$$

For evaluating H_2, we have

$$H_2 = \frac{1}{\pi} \int_{\pi/n}^{\pi} \Phi_x (u) \ \frac{\operatorname{Sin} n u}{u} \ du$$

and

$$| H_2 | \leq \int_{\pi/n}^{\pi} \frac{| \Phi_x (u) - \Phi_x (u + \pi /n) \mid \operatorname{Sin}^{\beta} \frac{u}{2}}{u. \operatorname{Sin}^{\beta} \frac{u}{2}} \ du$$

Applying Hölder's inequality, we have

$$| H_2 | \leq \left[\int_{\pi/n}^{\pi} | \Phi_x (u) - \Phi_x (u + \pi /n) |^p \operatorname{Sin}^{\beta p} \frac{u}{2} \ du \right]^{1/p}$$

$$\text{x} \left[\int_{\pi/n}^{\pi} \frac{du}{u^q \operatorname{Sin}^{\beta q} \frac{u}{2}} \right]^{1/q}$$

$$\leq \left[\int_{-\pi}^{\pi} | f(u) - f(u + \pi/n) |^{p} \operatorname{Sin}^{\beta p} \frac{u}{2} du \right]^{1/p}$$

$$\times \frac{1}{\operatorname{Sin}^{\beta} \frac{\pi}{2n}} \left[\int_{\pi/n}^{\pi} \frac{du}{u^{q}} \right]^{1/q} \text{[by mean value theorem]}$$

$= 0 \left(\psi \left(\frac{1}{n}\right) . n^{-\beta}\right) . \; 0 \; (n^{\beta+1/p})$ [by the definition of the class $W'(L^p, \psi(h))$]

$$= 0 \left(\psi \left(\frac{1}{n}\right) . \; n^{1/p}\right) .$$

which completes the proof of the theorem H.

REFERENCES

[1] **Shin Ichi, Izumi:** Notes on Fourier series (XXI) On the degree of approximation of partial sums of Fourier series J.L.M.S. (25), 1950, 40–42.

Department of Mathematices
Aligarh Muslim University
Aligarh - 202001 (INDIA)