A LA MEMOIRE D'ATATÜRK AU CENTENAIRE DE SA NAISSANCE

On The Homology Group of The Complex Analytic Manifolds

by

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DEDICATION TO ATATÜRK'S CENTENNIAL

Holding the torch that was lift by Atatürk in the hope of advancing our. Country to a modern level of civilization, we celebrate the one hundredth anniversary of his birth. We know that we can only achieve this level in the fields of science and technology that are the wealth of humanity by being productive and creative. As we thus proceed, we are conscious that, in the words of Atatürk, "the truest guide" is knowledge and science.

As members of the Faculty of Science at the University of Ankara we are making every effort to carry out scientific research, as well as to educate and train technicians, scientists, and graduates at every level. As long as we keep in our minds what Atatürk created for his Country, we can never be satisfied with what we have been able to achieve. Yet, the longing for truth, beauty, and a sense of responsibility toward our fellow human beings that he kindled within us gives us strength to strive for even more basic and meaningful service in the future.

From this year forward, we wish and aspire toward surpassing our past efforts, and with each coming year, to serve in greater measure the field of universal science and our own nation.

On The Homology Group of The Complex Analytic Manifolds

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SUMMARY

It is shown that the Homology group $F/[F, F]$ of the Complex Analytic Manifold X, where F is the Fundamental group of X and $[F, F]$ the Commutator subgroup, is isomorphic to the Cohomology group $H^o(X, A)$ of the structure restricted sheaf A of X. while $[F, F]$ determines A as the normal covering space of X, i. e., the *homology covering* space of X.

In this paper, we investigate therefore the relationship of the *Homology group of* the complex analytie manifold X to the Cohomology group $H^o(X,A)$ of its structure restricted sheaf A.

1. Complex Manifolds.

Definition 1.1. A complex manifold of dimension n is a connected Hausdorff space X together with a collection of charts $\{(U_\alpha, z_\alpha)\}$ with the following properties:

1. The U_α, $\alpha \in I$, form an open covering of X.

2. Each z_α is a homeomorphic mapping of U_α onto an open subset $B_\alpha \subset \mathbb{C}^n$.

3. If $U_\alpha \cap U_\beta \neq \varnothing$, then $z_{\alpha\beta} = z_\beta \circ z_\alpha^{-1}$ is topological on $z_\alpha (U_\alpha \cap U_\beta)$.

The collection of charts $\{(U_\alpha, z_\alpha)_{\alpha \varepsilon I}\}$ is called a complex atlas for X.

One speaks of a complex analytic manifold of dimension n as soon as the mappings $z_{\alpha\beta}$ are holomorphic. We also say that X is endowed with an analytic structure, A point $p \in U_\alpha$ being uniquely determined by $z_\alpha(p)$, z_α is referred to as a *local variable*, or *local parameter*, The subscript α is often dropped, and $z(p)$ is identified with p.

2. Covering Spaces.

Definitions. Let X and X* be complex manifolds of dimension n. Consider a mapping $\pi\colon X^* \to X$. π is said to be a local homeomorphism if every point $x^* \in X^*$ has a neighborhood U* such that the restriction of π to U* is a homeomorphism. When this is the case then the pair (X*, π) is called a covering manifold of X.

The point $x = \pi(x^*)$ is the projection of x*, while x* is said to lie over x. The mapping π is called a projection. π is continuous.

It is always possible to choose U* and $U = \pi(U^*)$ within the domains of local parameters z* and z respectively. It is convenient to identify U* and U with their images the local parameters z* and z respectively, and use the notation $z = \pi(z^*)$ for the projection map.

If X is a complex analytic manifold with atlas $\{(U_\alpha, z_\alpha)\}$, then X* can be endowed with a unique complex analytic structure which makes the mapping $\pi\colon X^* \to X$ holomorphic as follows: We require the atlas $\{(U^*_\beta, z^*_\beta)\}$ to be such that π is one to one on U_β^* and the functions $z_\alpha \circ \pi \circ z_\beta^{*-1}$ holomorphic whenever they are defined. In other words, the complex analytic structure on X* is such that $f \circ \pi$ is holomorphic on X* as soon as f is holomorphic on X.

Note: If X and X* are merely topological spaces, then (X*, π) is simply called a covering space of X. In the sequel, X is always a complex analytic manifold with fundamental group $F \neq 1$.

Definition 2.1. Let f be defined on X, and $x_0 \in X$. f is called holomorphic at x_0 if there exists an $\alpha \in I$ and a neighborhood $U(x_0) \subset X \cap U_\alpha$ such that $f \circ z_\alpha^{-1}$ is holomorphic on $z_\alpha(U) \subset B_\alpha$. f is called holomorphic on X if f is holomorphic at every point $x \in X$.

The totality of holomorphic functions on X is denoted by A(X). It is a ring (or C-algebra).

3. Restricted Sheaf.

We shall make A(X) into a covering topological space A of X as follows:

Definitions. Let $f \in A(X)$, and $x \in X$ a point. f can be expanded into a power series f_x convergent at z, the local parameter of x. The

totality of such power series at x as f runs through A(X) is denoted by A_x which is again a ring (C-algebra) isomorphic to A(X). The disjoint union

$$A = \bigvee_{x \in X} A_x$$

is a set over X with a natural projection

$$\pi: A \to X$$

mapping each f_x onto the point of expansion x.

We introduce on A a natural topology as follows [1]:

Let $f_{x_0} \in A$. Then there exists an open neighborhood $U = U(x_0)$ $\subset X$ and a holomorphic function $f \in A(X)$ such that f_{x_0} converges uniformly to $f|U$ in U where U is at the same time the local parameter. Therefore the function f can be expanded in a convergent power series at each point $x \in U$. Hence $f \in A(X)$ induces a mapping

$$s: U \to A$$

defined by $s(x) = f_x \in A$, $x \in U$, such that $\pi \circ s = 1_U$ and $s(x_0) = f_{x_0}$ $\in s(U) \subset A$.

All such sets s(U) form a system of neighborhoods of $f_{x_0} \in A$ which induces a topology in A.

In this topology s is continuous. Indeed, let $\sigma \in A$. By construction of A, there exists $x \in X$ such that $\sigma \in A_x$. Hence there exists $f \in A(X)$ such that $f_x = \sigma$. If s(U) is any neighborhood of σ, then there exists $V(x) \subset U$ such that $s(V) \subset s(U)$. Similarly, π is continuous in this topology.

s is called a section over U, and the totality of sections over U is denoted by $\Gamma(U, A)$. The definition holds for any open set $U \subset X$.

Theorem 3.1. Let $s \in \Gamma(U, A)$. Then $\pi: s(U) \to U$ is topological, and $s = (\pi|s(U))^{-1}$.

Proof. For $x \in U$

so $(\pi|s(U))(s(x)) = s \circ \pi \circ s(x) = 1_U \circ s(x) = s(x)$.

Hence

$$so (\pi|s(U)) = 1_{s(U)}.$$

We shall be mainly interested in global sections, i.e., sections over X.

It is shown that $A(X) \simeq \Gamma(X, A)$ [1]. Let $A(W)$ be the set of holomorphic functions on X restricted to W, and $\gamma_{w, v}: A(W) \to A(V)$ the usual restriction mapping. Then $\{A(W), \gamma_{w,v}\}$ is a pre-sheaf. The corresponding sheaf is just the restricted sheaf A. We have $A(W) \simeq \Gamma(W, A)$ which can be extended to X.

If $A'(X)$ is a subring of $A(X)$, then a similar construction yields a restricted subsheaf $A' \subset A$. Also $A'(X) \simeq \Gamma(X, A')$ where $\Gamma(X, A') \subset \Gamma(X, A)$ is a subgroup.

Definition 3.1. The topological space A is called the restricted sheaf of convergent power series called germs of holomorphic functions f ∈ A(G).

4. Characteristic Features of A.

1. Every section over an open set $U \subset X$ can be extended to a global section over X.

2. The points over x form the ring (C-algebra) $A_x = \pi^{-1}(x)$ of germs at x called a stalk of the restricted sheaf A. Any two stalks are isomorphic with each other. Thereby every stalk is isomorphic to A(X).

3. Let $s_1, s_2 \in \Gamma(X, A)$, and $x_0 \in X$. If $s_1(x_0) = s_2(x_0)$ then $s_1 = s_2$ over the whole X. Indeed, $s_1(x_0) = s_2(x_0)$ implies that $s_1(x) = s_2(x)$ on some open neighborhood $U(x_0) \subset X$. Hence if $f_1, f_2 \in A(X)$ are the corresponding holomorphic functions then $f_1 = f_2$ over $U(x_0)$ and thereby over the whole X.

Definition 4.1. A *sheaf isomorphism* is a topological stalk preserving mapping of A onto itself. It is called a cover transformation of A.

In conclusion A is a covering space of X with the sheaf isomorphisms as cover transformations. It has the following properties:

5. Properties of A as a covering space of X.

Let A_{x_0} be a stalk of A and $\sigma \in A_{x_0}$. Then to each point σ of A_{x_0} there corresponds a unique section $s \in \Gamma(X, A)$ such that $s(x_0) = \sigma$. Thus there is a one to one correspondence between the points of Ax_0 and the sections of $\Gamma(X, A)$. We may state

Theorem 5.1. The group T of cover transformations of A is isomorphic to the abelian group $\Gamma(X, A)$ whose elements are uniquely determined by the points (germs) on A_{x_0}.

Hence T is transitive, i.e., given σ_1, $\sigma_2 \in A_{xo}$, there is a unique transformation which carries σ_1 into σ_2. In other words T transforms every point of A_{xo} to all the points of A_{xo}.

We conclude that A is a regular covering space of X, since its group T of covering transformations acts transitively over each point $\sigma \in A$. We shall see later that its fundamental group projects to a normal subgroup, the commutator subgroup of the fundamental group of X.

Moreover, every open set U within the domain of local variable z is in view of theorem 3.1 evenly covered by A, i.e., every component of the inverse image $\pi^{-1}(U)$ is homeomorphic to U. Since every point of X has an evenly covered neighborhood, then A is complete.

Thus A is a complete regular covering space of X [4].

So is every restricted subsheaf A'.

6. Fundamental Group and Normal subgroups.

An arc in X is a continuous mapping $\gamma(t): I \to X$ where $I = [0, 1]$ is the closed unit interval.

Let the arcs $\gamma_1, \gamma_2 \subset X$ have common end points, i.e.,

$$\gamma_1(0) = \gamma_2(0), \gamma_1(1) = \gamma_2(1).$$

A continuous mapping

$$\gamma(t, u): I \times I \to X$$

is called a *deformation* of γ_1 into γ_2 if

$$\gamma(t, 0) = \gamma_1, \gamma(t, 1) = \gamma_2.$$

When such a deformation exists, we say that γ_1 is *homotopic* to γ_2, and we write $\gamma_1 \sim \gamma_2$.

Let x_0 be a fixed point in X. If $\gamma(0) = \gamma(1) = x_0$, then $\gamma(t)$ is a closed arc in X that begins and end at x_0. In the sequel only closed arcs that begin and end at x_0 will be considered.

The relation $\gamma_1 \sim \gamma_2$ is an equivalence relation. The equivalence classes are called homotopy classes. We shall denote the homotopy class of γ by $\{\gamma\}$. All homotopy classes can be multiplied in the usual way and the operation of multiplication is well defined. The multiplicative unit is the homotopy class of the degenerate arc $\gamma(t) = x_0$. It will be denoted by 1. It is easy to see that:

1. Multiplication is associative.

2. Every homotopy class $\{\gamma\}$ has an inverse $\{\gamma^{-1}\}$ since $\{\gamma\}\;\{\gamma^{-1}\}=1$.

Here $\gamma^{-1}(t) = \gamma(1-t)$, i.e., it is γ traced in the reversed sense.

We conclude that the homotopy classes of closed arcs with initial point x_0 form a group. It is called the *fundamental group* of X with respect to x_0. However it is easy to see that, since X is connected the abstract group is independent of x_0, i.e., if x_1 is another fixed point in X then the fundamental group with respect to x_1 is isomophic to the one with respect to x_0. The fundamental group of X will be denoted by F.

Let γ be an arc on X, with initial point $\gamma(0) = x_0$. We say that the arc $\gamma^*: I \to A$ covers γ, or that γ can be lifted to γ^* or γ^* is a lift of γ or it is a continuation of γ if

$$\pi(\gamma^*) = \gamma \text{ for all } t \in I.$$

The initial point $\gamma^*(0) = \sigma \in A_{xo}$ lies over x_0.

Theorem 6.1. Every arc γ with initial point x_0, can be lifted to a unique γ^* from any initial point $\sigma \in A_{xo}$.

Proof. Let E be the set of all $\tau \in [0,1]$ such that arcs $\gamma[0,\tau]$ can be uniquely lifted to the arcs $\gamma^*[0, \tau]$ with the initial point $\sigma \in A_{xo}$. $E \neq \varnothing$, since $0 \in E$. Let $\tau \in E$. Since A is complete we can determine an evenly covered neighborhood $U \subset X$ of $\gamma(\tau)$ and choose ε so small that the arc $\gamma[\tau, \tau, + \varepsilon] \subset U$. The point $\gamma^*(\tau)$ belongs to a component U^* of $\pi^{-1}(U)$. Since $\pi: U^* \to U$ is topological there is a unique way of extending $\gamma^*[0, \tau]$ to $\gamma^*[\tau, \tau + \varepsilon]$. Hence E is relatively open. We can similarly prove that the complement of E is open. In conclusion $E = [0, 1]$.

Thus A is a complete regular covering space of X, such that if γ is any arc in X with initial point $x_0 \in X$, and any point on A_{xo} then γ can be lifted uniquely to γ^* in A with initial point σ over x_0.

Moreover γ^* lies in the section $s(X)$ through $\sigma = s(x)$, and is uniquely determined by $s(X)$. We may write $\gamma^* = s(\gamma)$ at $\sigma = s(x_0)$ with $\pi(\gamma^*) = \gamma$.

Let $\gamma \subset X$ be a closed arc with initial point x_0. If the continuation along γ is a closed arc with initial point $\sigma \in A_{xo}$, then the same is true, by

the *monodromy theorem*, for any closed arc which is homotopic to γ. This is also true for γ^{-1}, and if it holds for γ_1 and γ_2 then it holds for $\gamma_1\gamma_2$. It follows that the homotopy classes $\{\gamma\}$ with a closed continuation γ^* from σ form a subgroup $D \subset F$, which is uniquely determined by (A, σ). We may write uniquely $s(D) = D^*$ at $\sigma = s(x_0)$, D^* is isomorphic to D with $\pi(D^*) = D$.

Suppose now that D and D_1 are determined by (A, σ) and (A, σ_1) respectively, $\sigma, \sigma_1 \in A_{xo}$. Let us join σ to σ_1 by an arc $a^* \subset A$. $\pi(a^*) = a \subset X$ is a closed arc from x_0. A given closed arc $\gamma_1 \subset X$ from x_0 determines a closed arc from σ_1 if and only if a $\gamma_1 a^{-1}$ determines a closed arc from σ. Hence D_1 consists of all homotopy classes $\{a^{-1} \gamma_1 a\}$ with $\{\gamma_1\} \in D$. Namely, D_1 is a conjugate subgroup of D in F.

Conversely, if $D_1 = a^{-1} Da$, then D_1 corresponds to (A, σ_1) where σ_1 is the terminal point on A_{xo} of the continuation a^* along a from x_0, and D_1 is lifted uniquely to $s_1(D_1) = D^*_1$ at σ_1. Therefore if D is a normal subgroup of F, then $D_1 = D = a^{-1} Da$ with $D^*_1 \cong D$ and there is a one to one correspondence between the conjugate subgroups of D and the associated sections over X which form a subgroup $\Gamma(X, A')$ of $\Gamma(X, A)$. Therefore, if D is a normal subgroup of F, then $D = a^{-1} Da$ is invarient under the subgroup T' of cover transformations isomorphic to $\Gamma(X, A')$ determined by the sections defined by the closed arcs $a \notin D$ from x_0.

We conclude that

Theorem 6.2. The fundamental group of the restricted subsheaf A' defined by the presheaf $\{A'(W), \gamma_{W,V}\}$ projects on and is isomorphic to a normal subgroup D of F or its conjugate subgroup $a^{-1} Da$ in F.

As a consequence we can state

Theorem 6.3. $\Gamma(X, A') \cong F/D$.

Proof. Each coset $Da = aD$ of F/D determines uniquely $\sigma_1 \in A'_{xo} \subset A_{xo}$ and thereby the section through σ_1.

These regular subspaces A' determined by the normal subgroups of F are called *normal covering spaces* of X.

Now, if $D_1, D_2 \subset F$ are two normal subgroups and A_1', A_2' the corresponding restricted subsheaves then clearly $A_2', \supset A_1'$ if and only if $D_2 \subset D_1$.

Thus the collection of restricted subsheaves of A determined by the normal subgroups of F is partially ordered and every chain has an upper bound in the collection. Hence by Zorn's lemma there is a maximal resricted subsheaf which is A. It corresponds of course to the smallest normal sungroup D for which F/D is additive. Hence D = [F, F] is the commutator subgroup of F. We may therefore state

Theorem 6.4. $\Gamma (X, A) \cong F/[F, F]$.

F/[F, F] is called the *homology group* of X, while A is the *normal covering space* determined by [F, F]. A is called the *homology covering space*

Hence in view of the already known result [2, 3]

$$H^0 (X, A) \cong \Gamma (X, A)$$

we have

Theorem 6.5. The homology group F/[F, F] of X is isomorphic to the cohomology group $H^0 (X, A)$ of the structure restricted sheaf A determined by [F, F] as the normal covering space, i.e., the homology covering space of X.

Corollary. The homology group of X is isomorphic to the additive group A(X) of holomorphic functions on X.

Corollary. Two analytic manifolds are holomorphically equivalent if and only if their A- sheaves are isomorphically equivalent, i.e., have the same fundamental group.

ÖZET

Gösteriliyor ki, X kompleks analitik manifoldunun F/[F, F] homoloji grubu, X in strüktürü A tahditli demetinin $H^0 (X, A)$ kohomoloji grubuna izomorftur. Aynı zamanda, A, [F, F] tarafından belirtilen ve homoloji örtü uzayı denilen, X in normal örtü uzayıdır.

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