COMMUNICATIONS

DE LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ D'ANKARA

Série A_1: Mathématiques

TOME : 31 ANNÉE : 1982

Radii Of p-Valence Of Certain Analytic Functions With Negative Coefficients

by

DASHRATH and VINOD KUMAR

17

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie

Communications de la Faculté des Sciences
de l'Université d'Ankara

La Revue "Communications de la Faculté des Sciences de l'Université d'Ankara" est un organe de publication englobant toutes les diciplines scientifique représentées à la Faculté des Sciences de l'Université d'Ankara.

La Revue, jusqu'à 1975 à l'exception des tomes I, II, III etait composé de trois séries

Série A : Mathématiques, Physique et Astronomie,
Série B : Chimie,
Série C : Sciences Naturelles.

A partir de 1975 la Revue comprend sept séries:

Série A_1 : Mathématiques,
Série A_2 : Physique,
Série A_3 : Astronomie,
Série B : Chimie,
Série C_1 : Géologie,
Série C_2 : Botanique,
Série C_3 : Zoologie.

En princice, la Revue est réservée aux mémoires originaux des membres de la Faculté des Sciences de l'Universiteé d'Ankara. Elle accepte cependant, dans la mesure de la place disponible les communications des auteurs étrangers. Les langues Allemande, Anglaise et Française seront acceptées indifféremment. Tout article doit être accompagnés d'un resume.

Les articles soumis pour publications doivent être remis en trois exemplaires dactylographiés et ne pas dépasser 25 pages des Communications, led dessins et figures portes sur les feuilles séparées devant pouvoir être reproduits sans modifications.

Les auteurs reçoivent 25 extraits sans couverture.

l'Adresse : Dergi Yayın Sekreteri,
Ankara Üniversitesi,
Fen Fakültesi,
Beşevler–Ankara

Radii Of p-Valence Of Certain Analytic Functions With Negative Coefficients

by

DASHRATH and VINOD KUMAR

(Received August 3, 1982; accepted December 31, 1982)

ABSTRACT

In this paper we determine the radii of p-valence of the function F(z) defined by

$$F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} z f'(z), \quad z \in D$$

where $D = \{z: |z| < 1\}$, $\lambda \geq 0$ and the function f (z) belongs to certain subclasses of analytic p-valent functions with negative coefficients.

1. INTRODUCTION

Let T_p denote the class of functions $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$

which are regular in the unit disc $D = \{z: |z| < 1\}$ and T_p^* denote that subclass of T_p whose members are p-valent in D. A function f(z) of T_p belongs to the class $T_p^*(A, B)$ if $zf'(z)/f(z)$ is subordinate to $p(1 + Az)/(1 + Bz)$, $z \in D$, where $-1 \leq A < B \leq 1$. Equivalently $f(z) \in T_p^*(A, B)$ if and only if there exists a function $\omega(z)$ regular in D and satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in D$, such that

$$(1.1) \qquad \frac{zf'(z)}{f(z)} = p\,\frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad z \in D$$

It follows from (1.1) that $f(z) \in T_p^*(A,B)$ if and only if

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \Big/ \left(\frac{Bzf'(z)}{f(z)} - Ap \right) \right| < 1, \quad z \in D.$$

Further f (z) is said to belong to the class C_p (A, B) if and only if $zf'(z)/p \in T_p^*(A, B)$. It is well known that the functions in $T_p^*(A, B)$ and $C_p(A, B)$ are p-valent starlike and p-valent convex respectively. Let $P_p^*(A, B)$ denote the class obtained by replacing $zf'(z)/f(z)$ by $f'(z)/z^{p-1}$ in the definition of $T_p^*(A, B)$. Clearly $f(z) \in P_p^*(A, B)$ implies $\operatorname{Re}\{f'(z)/z^{p-1}\} > 0$, and hence the functions in $P_p^*(A, B)$ are p-valent in D.

Recently Goel and Sohi [2] have established the following result for the class $T_p^*(A, B)$.

Theorem A. A function $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ belongs to $T_p^*(A, B)$ if and only if

$$(1.2) \qquad \sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \, |a_{n+p}| \leq (B-A)p.$$

The result is sharp with the extremal function

$$(1.3) \qquad f(z) = z^p - \sum_{n=1}^{\infty} \frac{(B-A)p}{(1+B)n + (B-A)p} z^{n+p}.$$

By using this result Goel and Sohi [2] obtained distortion and covering theorems and some other results for the classes $T_p^*(A, B)$ and $C_p(A, B)$. In the present paper we obtain some new results with the help of above theorem. Before using it we point out that the above theorem is valid only when $B \geq 0$. In fact in its proof Goel and Sohi [2] used the inequality

$$\left| \sum_{n=1}^{\infty} -n \, |a_{n+p}| z^{n+p} \right| - \left| (B-A)pz^p - \sum_{n=1}^{\infty} [nB + (B-A)p] \, |a_{n+p}| z^{n+p} \right|$$

$$\leq \sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \, |a_{n+p}| - (B-A)p.$$

We find that the above inequality holds only for $B \geq 0$. Since all the results except that of Theorem 2 of Goel and Sohi [2] have been obtained by using Theorem A, it is obvious that these are also valid only for $B \geq 0$.

Further we claim that the function f (z) given by (1.3) is not an extremal function for the purpose, since, in (1.2), equality does not hold for it. In fact for such a f (z)

$$\sum_{n=1}^{\infty} [(1+B)n+(B-A)p] \, | a_{n+p} |$$

$$= \sum_{n=1}^{\infty} [(1+B)n+(B-A)p] \left[\frac{(B-A)p}{(1+B)n+(B-A)p} \right]$$

$$= \infty$$

$$\neq (B-A)p.$$

We suggest that the function f (z) given by

$$f(z) = z^p - \frac{(B-A)p}{(1+B)n+(B-A)p} z^{n+p}$$

is a suitable extremal function, since, the equality holds in (1.2) for it.

We also need the following result for the class P_p^* (A, B), which is due to Shukla and Dashrath [3].

Theorem B. A function $f(z) = z^p - \sum_{n=1}^{\infty} | a_{n+p} | z^{n+p}$ belongs to P_p^* (A, B), $B \geq 0$, if and only if

$$\sum_{n=1}^{\infty} (n+p)(1+B) \, |a_{n+p}| \leq (B-A)p. \tag{1.4}$$

The result is sharp.

In this paper we determine the radius of p-valence of the function

$$F(z) = (1-\lambda) \, f(z) + \frac{\lambda}{p} z f'(z), \; \lambda \geq 0,$$

under the assumption that $B \geq 0$, when f (z) is in T_p^* (A, B), C_p (A, B) or P_p^* (A, B). All the results are sharp and generalize the recent results of Bhoosnurmath and Swamy [1].

Throughout this paper we assume that $B \geq 0$ and $\lambda \geq 0$.

2. MAIN RESULTS

Theorem 1. If $f(z) = z^p - \sum_{n=1}^{\infty} | a_{n+p} | z^{n+p} \in T_p^*$, then

$$\sum_{n=1}^{\infty} (n+p) \mid a_{n+p} \mid \leq p.$$

Proof. Suppose $\sum_{n=1}^{\infty} (n+p) \mid a_{n+p} \mid = p + \varepsilon$, where $\varepsilon > o$.

Then there exists an integer N such that

$$\sum_{n=1}^{N} (n+p) \mid a_{n+p} \mid > p + \frac{\varepsilon}{2}.$$

For z in the interval $[p/(p+\varepsilon/2)]^{\frac{1}{N}} < z < 1$, we have

$$G(z) = \frac{f'(z)}{z^{p-1}} \leq p - \sum_{n=1}^{N} (n+p) \mid a_{n+p} \mid z^n$$

$$\leq p - z^N \sum_{n=1}^{N} (n+p) \mid a_{n+p} \mid$$

$$< p - (p + \varepsilon/2) z^N$$

$$< 0.$$

Since $G(0) > 0$, there exists a real number z_0, $0 < z_0 < 1$, for which

$G(z_0) = \dfrac{f'(z_0)}{{z_0}^{p-1}} = 0$. But this is contrary to the fact that f (z) is p-valent in D. Hence the required result follows.

Remark. For $p = 1$, our theorem generalizes Theorem 3 of Silverman [4].

Corollary 1. $T_p{}^* = T_p{}^* (-1, 1) = P_p{}^* (-1,1)$.

Theorem 2. Let $f(z) \in T_p{}^* (A, B)$ and $F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$ for $z \in D$. Then F (z) is p-valently starlike of order δ, $0 < \delta < 1$, for $|z| < r(p, \lambda, \delta, A, B)$, where

$$r(p, \lambda, \delta, A,B) = \inf_n \left[\frac{\{(1+B)n + (B-A)p\}(1-\delta)p}{(B-A)\{n+p(1-\delta)\}(p+n\lambda)}\right]^{\frac{1}{n}} \quad n = 1, 2, 3, \ldots .$$

The result is sharp.

Proof. We have.

$$F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$$

$$= z^p - \sum_{n=1}^{\infty} \left(\frac{p+n\lambda}{p}\right) |a_{n+p}| z^{n+p}.$$

Now it suffices to show that the values of $\frac{zF'(z)}{F(z)}$ lie in a circle centered at p whose radius is p (1 – δ) for $|z| < r(p, \lambda, \delta, A, B)$.

We have

$$\left|\frac{zF'(z)}{F(z)} - p\right| = \left|\frac{-\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right) |a_{n+p}| z^n}{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right) |a_{n+p}| z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right) |a_{n+p}| |z|^n}{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right) |a_{n+p}| |z|^n}.$$

The last expression is bounded above by p(1 – δ) if

$$\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right) |a_{n+p}| |z|^n \leq p(1-\delta)\{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right) |a_{n+p}| |z|^n\}$$

or if

$$(2.1) \quad \sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)\left(\frac{n+p(1-\delta)}{1-\delta}\right) |a_{n+p}| |z|^n \leq p.$$

Since $f(z) \in T_p^*(A, B)$, we have from (1.2)

$$\sum_{n=1}^{\infty}\left[\frac{(1+B)n+(B-A)p}{B-A}\right] |a_{n+p}| \leq p.$$

Hence (2.1) holds if

$$\left(\frac{p+n\lambda}{p}\right)\left(\frac{n+p(1-\delta)}{1-\delta}\right)|a_{n+p}|\ |z|^n \le \left[\frac{(1+B)n+(B-A)}{B-A}\right]|a_{n+p}|$$

or if

$$|z| \le \left[\frac{\{(1+B)n+(B-A)p\}\,p(1-\delta)}{(B-A)(p+n\lambda)\ \{n+p(1-\delta)\}}\right]^{\frac{1}{n}}, \ n = 1, 2, 3, \ldots.$$

The result is sharp for the function

$$f(z) = z^p - \frac{p(B-A)}{(1+B)n+(B-A)p} z^{n+p}, \ n = 1,2,3, \ldots \ .$$

Corollary 2.1. Let $f(z) \in T_p{}^*$ and $F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$ for $z \in D$. Then $F(z)$ is p-valently starlike of order δ, $0 \le \delta < 1$, for $|z| < r(p, \lambda, \delta, -1, 1)$ where

$$r(p, \lambda, \delta, -1, 1) = \inf_n \left[\frac{p(n+p)\ (1-\delta)}{(p+n\lambda)\ \{n+p(1-\delta)\}}\right]^{\frac{1}{n}}, \ n=1, 2, 3, ..$$

The result is sharp.

Corollary 2.2. Let $f(z) \in T_p{}^*(A, B)$. Then $f(z)$ is p-valently starlike of order δ, $0 \le \delta < 1$, in

$$|z| < r(p,0,\delta,A,B) = \inf_n \left[\frac{\{(1+B)n+(B-A)p\}\,p(1-\delta)}{p(B-A)\ \{n+p(1-\delta)\}}\right]^{\frac{1}{n}}, \ n=1,2,3,\ldots$$

The result is sharp.

Corollary 2.3. Let $f(z) \in T_p{}^*(A, B)$. Then $f(z)$ is p-valently convex of order δ, $0 \le \delta < 1$ in

$$|z| < r(p,1,\delta,A,B) = \inf_n \left[\frac{\{(1+B)n+(B-A)p\}\,p(1-\delta)}{(p+n)(B-A)\ \{n+p(1-\delta)\}}\right]^{\frac{1}{n}}, n=1,2,3,\ldots$$

The result is sharp.

Corollary 2.4. Let $f(z) \varepsilon T_p{}^*(A,B)$ and $c > -p$, then

$F(z) = \frac{\{z^c f(z)\}'}{(p+c)\, z^{c-1}}$, for $z \in D$, is p-valently starlike of order δ, $o \le \delta < 1$, in

$$|z| < r\left(p, \frac{p}{p+c}, \delta, A, B\right) = \inf_{n} \left[\frac{\{(1+B)\,n + (B-A)\,p\}\,(1-\delta)(p+c)}{(B-A)(p+c+n)\{n+p\,(1-\delta)\}}\right]^{\frac{1}{n}}$$

$$n = 1, 2, 3, \ldots.$$

The result is sharp.

Theorem 3. Let $f(z) \in C_p(A, B)$ and $F(z) = (1-\lambda) f(z) + \frac{\lambda}{p} zf'(z)$ for $z \in D$. Then $F(z)$ is p-valently close-to-convex in D if $\lambda < \frac{1+B}{B-A}$ and $F(z)$ is p-valently convex of order δ, $0 \leq \delta < 1$, in $|z| < r(p, \lambda, \delta, A, B)$ where $r(p, \lambda, \delta, A, B)$ is as stated in Theorem 2. The result is sharp.

Proof. We have

$$F'(z) = (1-\lambda) f'(z) + \frac{\lambda}{p} \{zf'(z)\}'.$$

Therefore

$$\text{(2.2)} \quad \operatorname{Re}\left\{\frac{F'(z)}{f'(z)}\right\} = 1 - \lambda + \frac{\lambda}{p} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

Since $f(z) \in C_p(A, B)$, we can easily prove that

$$\text{(2.3)} \qquad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq p\,\frac{1+A}{1+B}.$$

By using (2.3) in (2.2) we have

$$\operatorname{Re}\left\{\frac{F'(z)}{f'(z)}\right\} \geq 1 - \lambda + \frac{\lambda}{p} \cdot p\,\frac{1+A}{1+B}$$

$$\geq 1 - \lambda + \lambda\,\frac{1-A}{1+B}.$$

Now $\operatorname{Re}\left\{\frac{F'(z)}{f'(z)}\right\} > 0$ if $1 - \lambda + \lambda\,\frac{1+A}{1+B} > 0$ or if $\lambda < \frac{1+B}{B-A}$.

Hence $F(z)$ is p-valently close-to-convex in D if $\lambda < \frac{1+B}{B-A}$.

We now prove that F (z) is p-valently convex of order $\delta, 0 < \delta < 1$ in $|z| < r(p, \lambda, \delta, A, B)$. We have

$$(2.4) \quad \frac{zF'(z)}{p} = (1-\lambda)\frac{zf'(z)}{p} + \frac{\lambda z}{p}\left\{\frac{zf'(z)}{p}\right\}' \text{ for } z \in D.$$

Since $f(z) \in C_p(A,B)$ it follows that $\dfrac{zf'(z)}{p} \in T_p^*(A,B)$

Applying Theorem 2 with $\dfrac{zf'(z)}{p}$ in place of f (z), it follows from

(2.4) that $\dfrac{zF'(z)}{p}$ is p-valently starlike of order δ in $|z| < r(p,\lambda,\delta, A,B)$,

equivalently, F (z) is p-valently convex of order δ in $|z| < r(p,\lambda,\varrho,A,B)$. The result is sharp for the function.

$$f(z) = z^p - \frac{p^2(B-A)}{(n+p)\{(1+B)n + (B-A)p\}} z^{n+p}, \quad n = 1,2,3,\ldots.$$

Theorem 4. Let $f(z) \in P_p^*(A, B)$ and $F(z) = (1-\lambda) f(z) + \dfrac{\lambda}{p} zf'(z)$ for $z \in D$. Then $\mathrm{Re}\left\{\dfrac{F'(z)}{z^{p-1}}\right\} > p\delta, \; 0 \leq \delta < 1$ for $|z| < r(p, \lambda, \delta, A, B)$, where

$$r(p, \lambda, \delta, A, B) = \inf_n \left[\frac{p(1+B)(1-\delta)}{(p+n\lambda)(B-A)}\right]^{\frac{1}{n}}, \quad n = 1, 2, 3, \ldots.$$

The result is sharp.

Proof. To prove the result it is sufficient to show that the values of $\dfrac{F'(z)}{z^{p-1}}$ lie in a circle centered at p whose radius is $p(1-\delta)$ for

$|z| < r(p, \lambda, \delta, A, B)$. We have

$$\left|\frac{F'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty} \frac{(n+p)(p+n\lambda)}{p} |a_{n+p}| z^n\right|$$

$$\leq \sum_{n=1}^{\infty} \frac{(n+p)(p+n\lambda)}{p} |a_{n+p}| |z|^n.$$

Hence $\left| \dfrac{F'(z)}{z^{p-1}} - p \right| \leq p(1-\delta)$ if

$$\sum_{n=1}^{\infty} \left\{ \frac{(n+p)(p+n\lambda)}{p(1-\delta)} \right\} |a_{n+p}| \; |z|^n \leq p.$$

Since $f(z) \in P_p^*(A, B)$, we have from (1.4)

$$\sum_{n=1}^{\infty} \left\{ \frac{(n+p)(1+B)}{B-A} \right\} |a_{n+p}| \leq p.$$

Now proceeding as in the proof of Theorem 2, we can easily obtain the required result.

The result is sharp for the function

$$f(z) = z^p - \frac{p(B-A)}{(1+B)(n+p)} z^{n+p}, \; n = 1, 2, 3, \ldots \; .$$

Remark: Putting $p = 1$ and taking $A = (2\alpha - 1)$, $B = 1$, where $0 \leq \alpha < 1$, in the above theorems we get the results obtained by Bhoosnurmath and Swamy [1].

ACKNOWLEDGEMENT

The authors wish to thank Dr. S.L. Shukla for his kind encouragement.

REFERENCES

[1] **Bhoosnurmath, Subhas S. and Swamy S.R.:** Analytic functions with negative coefficients, Indian J. Pure Appl. Math., 12 (6), 1981, 738-742.

[2] **Goel, R.M. and Sohi, N.S.:** Multivalent functions with negative coefficients, Indian J. Pure Appl. Math., 12 (7), 1981, 844-853.

[3] **Shukla, S.L. and Dashrath:** On certain classes of multivalent functions with negative coefficients, Soochaw J. Math., 8, (1982), 179 – 188.

[4] **Silverman, H.:** Univalent functions with negative coefficients, Proc. Am. Math. Soc., 51 (1975), 109-116.

Department of Mathematics
Janta College
Bakewar-206124
Etawah (U.P.)
India.