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On Generalized Mean Values Of An Entire Dirichlet Series

by

G. S. SRIVASTAVA and N. KUMAR

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On Generalized Mean Values Of An Entire Dirichlet Series

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ABSTRACT

For the entire function represented by an everywhere convergent Dirichlet series having (p,q)–order ρ and lower (p,q)–order λ, we have defined the generalized mean values $m_{\delta,k}(\sigma)$ as:

$$m_{\delta,k}(\sigma) = \frac{1}{(\log^{[q-1]}\sigma)^k} \int_a^\sigma \frac{M_\delta(x)\ (\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\, dx,$$

where $0 < \delta < \infty$, $a = \exp^{[q-1]}(0)$, $\Lambda_{[q]}(x) = \prod_{i=0}^{q} \log^{[i]}x$, $k \in R$ (R is the field of reals),

$M_\delta(x) = \log^{[p-2]} I_\delta(x)$,

$$I_\delta(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |f(\sigma+it)|^\delta\, dt$$ and p and q are integers sucb that $p \geq q+1 \geq 0$.

In this paper, we have obtained some growth properties of $m_{\delta,k}(\sigma)$, which include entire functions of zero as well as of infinite order. Beside proving the asymptotic relation between $I_\delta(\sigma)$ and $m_{\delta,k}(\sigma)$ we have also studied the growth properties of means of more than one entire function. The results that we obtain here generalize and improve several known results.

1. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, ($s = \sigma + it$, $\lambda_{n+1} > \lambda_n$, $\lambda_n \mapsto \infty$ as $n \mapsto \infty$), be an entire Dirichlet series whose exponents are subject to the condition $\lim\limits_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$.

The (p,q) - order ρ (p,q) and lower (p,q) - order λ (p,q) of f (s) are defined as:

$$(1.1) \qquad \lim_{\sigma\mapsto\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p]}M(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho(p,q) \equiv \rho\\ \lambda(p,q) \equiv \lambda\end{matrix}$$

where p and q are integers such that $p \geq q+1 \geq 0$, $M(\sigma) = \text{l.u.b}_{-\infty \leq t \leq \infty} |f(\sigma+it)|$, $\log^{[q]}x = \log\log\ldots\log$ (q times) x and $\log^{[0]}x = x$. For further details we refer to [7].

For an entire function f (s) having (p,q) – order ρ and lower (p,q) –order λ we define the generalized mean values

$$(1.2) \quad m_{\delta,k}(\sigma) = \frac{1}{(\log^{[q-1]}\sigma)^k} \int_a^\sigma \frac{M_\delta(x)\,(\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\, dx,$$

where $0<\delta<\infty$, $a = \exp^{[q-1]}(0)$, $\Lambda_{[q]}(x) = \prod_{i=0}^{q} \log^{[i]}x$, $k \in R$ (R is the field of reals), $M_\delta(x) = \log^{[p-2]} I_\delta(x)$ and

$$I_\delta(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^\delta\, dt.$$

The growth properties of the means $m_{\delta,k}(\sigma)$ for $p = 2$, $q = 0$ and $p = 2$, $q = 1$ have been studied in great details. Juneja [6], Bajpai [2] and others obtained the order relations for $I_\delta(\sigma)$ and $m_{\delta,k}(\sigma)$ for entire functions of finite order having index pair (2,0).

The aim of this paper is to obtain some results for these mean values in the general case of entire function having (p,q) – order ρ, which include entire functions of zero order as well as of infinite order, for which the results of Gupta and Shakti Bala ([5], Th. 3), Bajpai ([2], p. 32) and Juneja ([6], p. 310) do not hold. To avoid the trivial cases we shall assume that f (s) is not an exponential polynomial.

2. THEOREM 1. Let f (s) be an entire function of (p,q) –order ρ and lower (p,q) –order λ, then

$$(2.1) \quad \lim_{\sigma \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[2]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{\sigma \to \infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[2]} M_\delta(\sigma)}{\log^{[q]}\sigma}$$

PROOF. For $t = \exp^{[q-2]}\{(\log^{[q-2]}\sigma)^2\}$, from the definition of $m_{\delta,k}(\sigma)$, we have

$$m_{\delta,k}(t) \geq \frac{1}{(\log^{[q-1]}t)^k} \int_\sigma^t \frac{M_\delta(x)\,(\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\, dx.$$

Since $I_\delta(\sigma)$ is an increasing function of σ therefore

$\log^{[p-2]} I_\delta(\sigma) = M_\delta(\sigma)$ will also be an increasing function of σ.
Thus

$$m_{\delta,k}(t) \geq \frac{M_\delta(\sigma)}{(\log^{[q-1]} t)^k} \int_\sigma^t \frac{(\log^{[q-1]} x)^{k-1}}{\Lambda_{[q-2]}(x)}\, dx$$

$$= \frac{M_\delta(\sigma)}{(\log^{[q-1]} t)^k} \left[\frac{(\log^{[q-1]} t)^k - (\log^{[q-1]} \sigma)^k}{k}\right]$$

$$= \frac{M_\delta(\sigma)}{k}\left[1 - \frac{1}{2^k}\right].$$

Thus proceeding to the limits as $\sigma \mapsto \infty$, we get

$$(2.2)\quad \lim_{\sigma \mapsto \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[2]} m_{\delta,k}(t)}{\log^{[q]} t} \geq \lim_{\sigma \mapsto \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[2]} M_\delta(\sigma)}{\log^{[q]} \sigma} = \begin{matrix}\rho\\ \lambda\end{matrix},$$

since $\sigma \mapsto \infty$ implies $t \mapsto \infty$ and $\log^{[q]} t \simeq \log^{[q]} \sigma$ and from ([4], Th.1)

$$\lim_{\sigma \mapsto \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[q]} I_\delta(\sigma)}{\log^{[q]} \sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}.$$

Again from (1.2), we have

$$m_{\delta,k}(\sigma) \leq \frac{M_\delta(\sigma)}{(\log^{[q-1]} \sigma)^k} \left[\frac{(\log^{[q-1]} \sigma)^k - (\log^{[q-1]} \sigma_0)^k}{k}\right], \quad \sigma > \sigma_0^*,$$

which gives

$$(2.3)\quad \lim_{\sigma \mapsto \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[2]} m_{\delta,k}(\sigma)}{\log^{[q]} \sigma} \leq \begin{matrix}\rho\\ \lambda\end{matrix}.$$

On combining (2.2) and (2.3), we get (2.1).

REMARKS (i) For $(p,q) = (2,0)$ and $\delta \in Z_+$(the set of positive integers) Theorem 1 was proved by Gupta and Shakti Bala ([5], Th.1).

(ii) For $\delta = 2$ and index pair (2,0), from (2.1), we have

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$^*\sigma_0$ need not be same at each occurrence.

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[2]} m_{2,k}(\sigma)}{\sigma} = \begin{matrix}\rho \\ \lambda\end{matrix},$$

a result which was proved by Juneja ([6]. Th. 3) for $o<\rho<\infty$. The above result was also proved by Kamthan ([8], Lemma 1) under certain restriction on a_n's.

(iii) Theorem 1 generalizes and improves upon the result of Bajpai ([2], p. 32) also which he proved for index pair (2,0), $0<k<\infty$ and finite ρ.

(iv) The left hand equality of this theorem is due to Bose and Srivastava ([3], p. 16) for $p = 2$, $q = 0$, $\delta\geq 1$ and positive real k.

THEOREM 2. Let f (s) be an entire function of (p,q) –order ρ and lower (p,q) –order λ. Then

$$\text{(2.4)} \quad \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \left[\frac{M_\delta(\sigma)}{m_{\delta,k}(\sigma)}\right]^{1/\log^{[q]}\sigma} = \begin{matrix}\exp(\rho) \\ \exp(\lambda)\end{matrix},$$

Proof of this theorem is based on the following lemmas.

LEMMA 1. If $\log g(\sigma)$ is an indefinitely increasing convex function of σ $(\sigma>\sigma_0)$, then $\log^{[n]} g(\sigma)$ is also an indefinitely increasing convex function of σ $(\sigma>\sigma_0)$, where n is any positive integer.

PROOF. We prove this lemma by method of induction. By hypothesis Lemma 1 is true for $n = 1$. Now

$$\frac{d^2(\log g(\sigma))}{d\sigma^2} = \frac{g''(\sigma)\, g(\sigma) - (g'(\sigma))^2}{(g(\sigma))^2},$$

where dashes denote the differential coefficients with respect to σ. By assumption, the left hand side of the above relation is positive, and hence

$$\text{(2.5)} \qquad g''(\sigma)\, g(\sigma) - (g'(\sigma))^2 > 0 .$$

Further,

$$\frac{d^2(\log^{[2]} g(\sigma))}{d\sigma^2} = \frac{g''(\sigma)g(\sigma) + (g'(\sigma))^2[1+(\log g(\sigma))^{-1}]}{(g(\sigma))^2 \log g(\sigma)} .$$

Since $\log g(\sigma) \to \infty$ as $\sigma \to \infty$, therefore, using (2.5), we have for $\sigma > \sigma_0$,

$$\frac{d^2(\log^{[2]} g(\sigma))}{d\sigma^2} > 0.$$

Similarly, assuming the convexity of $\log^{[n-1]} g(\sigma)$ with respect to σ, we can show that $\log^{[n]} g(\sigma)$ will also be a convex function of σ for all large σ. This proves Lemma 1.

It is known [9] that $\log I_\delta(\sigma)$ is a convex function of σ. Hence $\log M_\delta(\sigma) = \log^{[p-1]} I_\delta(\sigma)$ is also a convex function of σ $(\sigma > \sigma_0)$.

COROLLARY. If $\log g(\sigma)$ is an indefinitely increasing convex function of $\log(\sigma)$ $(\sigma > \sigma_0)$, then $\log^{[n]} g(\sigma)$ is also an indefinitely increasing convex function of σ for $\sigma > \sigma_0$, where n is a positive integer.

This corollary may be proved in a way similar to above lemma.

LEMMA 2. $M_\delta(\sigma)/m_{\delta,k}(\sigma)$ is an increasing function of σ for large σ.

PROOF. We have

$$\frac{d[(\log^{[q-1]}\sigma)^k M_\delta(\sigma)]}{d[(\log^{[q-1]}\sigma)^k m_{\delta,k}(\sigma)]} = \frac{k(\log^{[q-1]}\sigma)^{k-1} M_\delta(\sigma) + M'_\delta(\sigma)(\log^{[q-1]}\sigma)^k \Lambda_{[q-2]}(\sigma)}{M_\delta(\sigma)\ (\log^{[q-1]}\sigma)^{k-1}}$$

$$= k + \frac{M'_\delta(\sigma)}{M_\delta(\sigma)} \Lambda_{[q-1]}(\sigma),$$

which increases as σ increases, since $\log M_\delta(\sigma)$ is an increasing convex function of σ. Hence Lemma 2 follows

PROOF OF THEOREM 2. Since

$$\frac{d}{d\sigma}\{\log m_{\delta,k}(\sigma)\} = \left(\frac{M_\delta(\sigma)}{m_{\delta,k}(\sigma)} - k\right)\frac{1}{\Lambda_{[q-1]}(\sigma)},$$

therefore,

$$\text{(2.6)}\quad \log m_{\delta,k}(\sigma) - \log m_{\delta,k}(\sigma_0) = \int_{\sigma_0}^{\sigma} N_{\delta,k}(x)\frac{dx}{\Lambda_{[q-1]}(x)},$$

where

$$\text{(2.7)}\quad N_{\delta,k}(x) = \frac{M_\delta(x)}{m_{\delta,k}(x)} - k.$$

By last lemma $N_{\delta,k}(\sigma)$ increases with σ, Hence we have

$$\log m_{\delta,k}(\sigma) - \log m_{\delta,k}(\sigma_0) < N_\delta(\sigma)\,(\log^{[q]}\sigma - \log^{[q]}\sigma_0).$$

Thus, using Theorem 1, we get

$$(2.8)\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[2]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{\log^{[q]}\sigma}.$$

Again, from (2.6), we get

$$\log m_{\delta,k}(t) - \log m_{\delta,k}(\sigma_0) \geq \int_\sigma^t N_{\delta,k}(x)\,\frac{dx}{\Lambda_{[q-1]}(x)}$$

$$\geq N_{\delta,k}(\sigma)\log 2,$$

where

$$t = \exp^{[q-2]}\{(\log^{[q-2]}\sigma)^2\},$$

which gives

$$(2.9)\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{\log^{[q]}\sigma} \leq \begin{matrix}\rho\\ \lambda\end{matrix}.$$

(2.7) implies

$$(2.10)\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log N_{\delta,k}(\sigma)}{\log^{[q]}\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log (M_\delta(\sigma)/m_{\delta,k}(\sigma))}{\log^{[q]}\sigma}$$

Combining (2.8), (2.9) and (2.10), we get (2.4).

This proves Theorem 2.

REMARKS (v) For index pair (2,0) and $\delta \in Z_+$ this theorem is due to Gupta and Shakti Bala ([5], Th. 2).

(vi) Theorem 2 was also proved by Kamthan ([8]. Th. 2) for $\delta = 2$ and $(p,q) = (2,0)$ under certain restriction on the coefficients.

Thus Theorem 2 generalizes and improves upon the results in [5] and [8].

THEOREM 3. If $0 < \lambda, \rho < \infty$ is satisfied, them

$$(2.11)\quad \log M_\delta(\sigma) \simeq \log m_{\delta,k}(\sigma) \text{ as } \sigma\to\infty.$$

PROOF. Let $0 < \lambda, \rho < \infty$. From (2.4), for arbitrary $\varepsilon > 0$ and all $\sigma > \sigma_0$, we have

$(\lambda-\varepsilon) \log^{[q]}\sigma < (\log M_\delta(\sigma) - \log m_{\delta,k}(\sigma)) < (\rho+\varepsilon) \log^{[q]}\sigma.$
This implies

$$\lim_{\sigma \mapsto \infty} \frac{\log M_\delta(\sigma)}{\log m_{\delta,k}(\sigma)} = 1 ,$$

in view of Theorem 1. Thus Theorem 3 follows.

REMARK. (vii) In particular for index pair (2,0) and $\delta \in Z_+$, Theorem 3 was proved by Gupta and Shakti Bala ([5], p.34) and Bajpai ([2]. p. 32] seperately.

THEOREM 4. Let f (s) be an entire function of (p,q)–order ρ and lower (p,q)–order λ. Then

$$(2.12) \quad \lim_{\sigma \mapsto \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log [m'_{\delta,k}(\sigma) / m_{\delta,k}(\sigma)]}{\log^{[q]}\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix} ,$$

where, $m'_{\delta,k}(\sigma)$ is the derivative of $m_{\delta,k}(\sigma)$ with respect to σ.

Proof of this theorem requires the following lemma.

LEMMA 3. $\log m_{\delta,k}(\sigma)$ is an increasing convex function of $\log^{[q]}\sigma$ for $\sigma > \sigma_0$.

PROOF. We have

$$\frac{d(\log m_{\delta,k}(\sigma))}{d(\log^{[q]}\sigma)} = \frac{M_\delta(\sigma) - k m_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)} = \frac{M_\delta(\sigma)}{m_{\delta,k}(\sigma)} - k,$$

which increases, since from Lemma 2, $M_{\delta,k}(\sigma) / m_{\delta,k}(\sigma)$ is an increasing function of σ $(\sigma > \sigma_0)$. This implies that

$$\frac{d^2(\log m_{\delta,k}(\sigma))}{d^2(\log^{[q]}\sigma)} > 0 \text{ for } \sigma > \sigma_0. \text{ Hence Lemma 3 follows.}$$

REMARK (viii) If we take $(p,q) = (2,0)$, $\delta = 2$ and $0 < k < \infty$ in Lemma 3 then Theorem 1 of [6] follows.

PROOF OF THEOREM 4. Lemma 3 implies that $\log m_{\delta,k}(\sigma)$ is differentiable almost everywhere with an increasing derivative, may be written as

$$\log m_{\delta,k}(\sigma) = 0(1) + \int_{\sigma_0}^{\sigma} \frac{m'_{\delta,k}(x)\, dx}{m_{\delta,k}(x)\, \Lambda_{[q-1]}(x)}, \quad \sigma > \sigma_0,$$

or,

$$\log m_{\delta,k}(\sigma) < 0(1) + \frac{m'_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)} \log^{[q]}\sigma,$$

which on using Theorem 1 gives

$$(2.13) \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[2]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

$$\leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\,[m'_{\delta,k}(\sigma) m_{\delta.k}(\sigma)\,]}{\log^{[q]}\sigma}.$$

Further, for $t = \exp^{[q-2]}\{(\log^{[q-2]}\sigma)^2\}$ we have

$$\log m_{\delta,k}(t) = \log m_{\delta,k}(\sigma) + \int_{\sigma}^{t} \frac{m'_{\delta,k}(x)\, dx}{m_{\delta,k}(x)\Lambda_{[q-1]}(x)}$$

$$> \frac{m'_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)} \log 2.$$

Since $\log^{[q]}\sigma \simeq \log^{[q]}t$ as $\sigma \to \infty$, therefore after some manipulations, above inequality gives

$$(2.14) \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\,[m'_{\delta,k}(\sigma)/m_{\delta,k}(\sigma)\,]}{\log^{[q]}\sigma} \leq \begin{matrix}\rho\\ \lambda\end{matrix}.$$

On combining (2.13) and (2.14) we get (2.12).

REMARKS (ix). If we take δ and k as in Remark (viii) then for index pair (2,0), (2.12) leads to a result; which was proved, respectively, by Agarwal ([1], Th. 1), and by Juneja ([6], p. 312) under the condition that $D = 0$.

(x) For $(p,q) = (2,1)$, $o < k < \infty$ and $D = 0$ Theorem 4 was proved by Vaish ([10],)h. 3).

THEOREM 5. Let $f_i(s) = \sum_{n=1}^{\infty} a_{i,n}\exp(s\lambda_{i,n})$ $(i = 1,2)$ be two entire functions of (p,q)–orders ρ_i and lower (p,q)–orders λ_i, respectively. Then if

(i) $\log^{[2]} m_{\delta,k}(\sigma) \simeq \log\left[\{\log m_{\delta,k}(\sigma,f_1)\}\{\log m_{\delta,k}(\sigma,f_2)\}\right]$,

the function $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ will be of (p,q)–order ρ and lower (p,q)–order λ such that

$$\rho_1 + \rho_2 \geq \rho \geq \lambda_1 + \lambda_2, \tag{2.15}$$

and if

(ii) $\log^{[2]} m_{\delta,k}(\sigma) \simeq \left[\{\log^{[2]} m_{\delta,k}(\sigma, f_1]\}\{\log^{[2]} m_{\delta,k}(\sigma,f_2)\}\right]^{1/2}$,
then

$$(\rho_1\rho_2)^{1/2} \geq \rho \geq \lambda \geq (\lambda_1\lambda_1)^{1/2}, \tag{2.16}$$

where $m_{\delta,k}(\sigma)$ and $m_{\delta,k}(\sigma,f_i)$ are mean values of $f(s)$ and $f_i(s)$, respectively.

PROOF. Applying Theorem 1 to the functions $f_1(s)$ and $f_2(s)$, we get

$$\frac{\log^{[2]} m_{\delta,k}(\sigma,f_1)}{\log^{[q]}\sigma} < \rho_1 + \frac{\varepsilon}{2} \tag{2.17}$$

and

$$\frac{\log^{[2]} m_{\delta,k}(\sigma,f_2)}{\log^{[q]}\sigma} < \rho_2 + \frac{\varepsilon}{2}, \tag{2.18}$$

for an arbitrary $\varepsilon > 0$ and $\sigma > \sigma_0$.

Adding (2.17) and (2.18).we have

$$\frac{\log\left[\log m_{\delta,k}(\sigma,f_1) \log m_{\delta,k}(\sigma,f_2)\right]}{\log^{[q]}\sigma} < \rho_1 + \rho_2 + \varepsilon. \tag{2.19}$$

Similarly, on proceeding for the limit inferior, we obtain

$$\frac{\log\left[\log m_{\delta,k}(\sigma,f_1) \log m_{\delta,k}(\sigma,f_2)\right]}{\log^{[q]}\sigma} > \lambda_1 + \lambda_2 - \varepsilon. \tag{2.20}$$

If hypothesis (i) holds, then from (2.19) and (2.20), we get

$$\lambda_1 + \lambda_2 - \varepsilon < \frac{\log^{[2]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} < \rho_1 + \rho_2 + \varepsilon,$$

for any $\varepsilon > 0$ and sufficiently large σ. Proceeding to the limits as $\sigma \to \infty$, it leads to (2.15).

Similarly, on multiplying (2.17) and (2.18) and then using hypothesis (ii) in place of (i), we get (2.16).

COROLLARY. If functions $f_1(s)$ and $f_2(s)$ are of regular (p,q)-growth, then f (s) will also be of regular (p,q)-growth such that

$$\rho = \rho_1 + \rho_2.$$

REMARKS (xi). If we take δ,k and (p,q) same as in Remark (viii) then Theorem 5 is due to Agarwal ([1], Th. 2).

(xii). If we take (p,q), k and D same as in Remark (x) then Theorem 2 due to Vaish [10] follows from Theorem 5.

THEOREM 6. If in Theorem 5, hypothesis (i) is replaced by

$$\text{(ia)} \quad \log \left\{ \frac{m'_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)} \right\} \simeq \log \left\{ \frac{m'_{\delta,k}(\sigma,f_1)\ m'_{\delta,k}(\sigma,f_2)}{m_{\delta,k}(\sigma,f_1)\ m_{\delta,k}(\sigma,f_2)} \right\},$$

then, we have

$$\rho_1 + \rho_2 \geq \rho \geq \lambda \geq \lambda_1 + \lambda_2.$$

And if hypothesis (ii) of Theorem 5 is replaced by

$$\text{(iia)} \quad \log \left\{ \frac{m'_{\delta,k}(\sigma)}{m_{\delta,k}(\sigma)} \right\} \simeq \left[\log \left\{ \frac{m'_{\delta,k}(\sigma,f_1)}{m_{\delta,k}(\sigma,f_1)} \right\} \log \left\{ \frac{m'_{\delta,k}(\sigma,f_2)}{m_{\delta,k}(\sigma,f_2)} \right\} \right]^{\frac{1}{2}}$$

then

$$(\rho_1\rho_2)^{1/2} \geq \rho \geq \lambda \geq (\lambda_1\lambda_2)^{1/2}.$$

PROOF. Instead of making use of Theorem 1, we use Theorem 4 for $f_1(s)$ and $f_2(s)$ and then proceed as in the proof of Theorem 5, and the results follow.

THEOREM 7. Both the results of Theorem 5 hold if the conditions (i) and (ii) of Theorem 6 are replaced by

$$\text{(ib)} \quad \log \left\{ \frac{M_\delta(\sigma)}{m_{\delta,k}(\sigma)} \right\} \simeq \log \left\{ \frac{M_\delta(\sigma,f_2)\, M_\delta(\sigma,f_1)}{m_{\delta,k}(\sigma,f_2)\, m_{\delta,k}(\sigma,f_1)} \right\}$$

and

$$\text{(iib)} \quad \log \left\{ \frac{M_\delta(\sigma)}{m_{\delta,k}(\sigma)} \right\} \simeq \left[\log \left\{ \frac{M_\delta(\sigma,f_1)}{m_{\delta,k}(\sigma,f_1)} \right\} \log \left\{ \frac{M_\delta(\sigma,f_2)}{m_{\delta,k}(\sigma,f_2)} \right\} \right]^{\frac{1}{2}},$$

respectively.

PROOF. This theorem can be proved using Theorem 2 on the lines of the Theorem 5.

NOTE (1). Corollary after Theorem 5 also holds for Theorems 6 and 7.

(2) Theorems 5, 6 and 7 may be easily extended to any finite number of entire functions.

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Department of Mathematics,
University of Roorkee,
Roorkee (U.P.), INDIA,