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On The λ-Type Of Analytic Functions Of Irregular Growth Defined By Dirichlet Series

by

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ABSTRACT

In this paper, for an analytic function f(s) in the half-plane Re s $< \alpha$, which is of irregular growth, it is shown that lower type is always zero and, therefore, to study precisely the growth of such analytic functions, the concept of λ-type has been introduced and then some relations which connect λ-type with the maximum term, have been obtained. In the last, formula for λ-type in terms of coefficients and exponents in Dirichlet series expansion for f(s) has been obtained.

1. Consider the Dirichlet series

$$(1.1) \qquad f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n),$$

where $\lambda_1 \geq 0, 0 < \lambda_n < \lambda_{n+1} \to \infty$, $s = \sigma + i\,t$(σ, t being real variables), $\{a_n\}^{\infty}$, is a sequence of complex numbers and

$$(1.2) \qquad \limsup_{n \to \infty} \frac{n}{\lambda_n} = D < \infty.$$

If the series given by (1.1) converges absolutely in the half-plane Re s $< \alpha$ ($-\infty < \alpha < \infty$), then it is known [3, P.166] that the series (1.1) represents an analytic function in Re s $< \alpha$, and since (1.2) holds we have

$$\alpha = -\limsup_{n \to \infty} \frac{\log |a_n|}{\lambda_n} .$$

Let D_α denote the class of all functions f(s) of the form (1.1) which satisfy (1.2) and are analytic in the half-plane Re s $< \alpha$ ($-\infty < \alpha < \infty$). Set

$$M(\sigma) \equiv M(\sigma, f) = \underset{-\infty < t < \infty}{\text{l.u.b.}} \; | f(\sigma + i\, t) |,$$

$$m(\sigma) \equiv m(\sigma, f) = \max_{n \geq 1} \; (|a_n| e^{\sigma \lambda_n})$$

and

$$N(\sigma) = \max\,(n : m(\sigma) = |a_n| \, e^{\sigma \lambda_n}$$

$M(\sigma)$, $m(\sigma)$ and $N(\sigma)$ are called maximum modulus, maximum term and the rank of maximum term respectively of f (s).

To study precisely the growth of analytic functions belonging to D_α, the concept of order ρ and lower order λ have been defined [1] as

$$(1.3) \qquad \lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\sigma)}{-\log \{1 - \exp(\sigma - \alpha)\}} = \begin{matrix} \rho \\ \lambda \end{matrix},$$

and then, it has been shown [1] that

$$(1.4) \qquad \lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log m(\sigma)}{-\log \{1 - \exp(\sigma - \alpha)\}} = \begin{matrix} \rho \\ \lambda \end{matrix} \quad (0 < \lambda \leq \rho < \infty)$$

where

$$(1.5) \qquad \liminf_{n \to \infty} (\lambda_{n+1} - \lambda_n) = \beta > 0.$$

Definition. f(s) *is said to be of reqular growth if* $\lambda = \rho$. *If* $\lambda < \rho$, *then it is said to be of irregular growth.*

Further, if f(s) of order ρ $(0 < \rho < \infty)$, type T and lower type t $(0 \leq t,\ T \leq \infty)$ of f(s) are defined as

$$(1.6) \qquad \lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M(\sigma)}{\{1 - \exp(\sigma - \alpha)\}^{-\rho}} = \begin{matrix} T \\ t \end{matrix},$$

and then Krishna Nandan [2] has obtained complete coefficient characterization for the type and lower type.

In this note, we first show that for a function of irregular growth, the lower type is always zero, and therefore for a function $f(s) \in D_\alpha$ of irregular growth, we introduce new growth parameters λ–type t_λ, and then obtain some relations which connect λ–type with the coefficients and exponents in the Dirichlet series expansion for f(s). Also we obtain some relations, involving type, λ–type and maximum term.

2. We need following lemmas in sequel.

Lemma 1. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the class* D_α, *with order* ρ, *lower order* λ $(0 \leq \lambda < \rho < \infty)$, *then*

$$(2.1) \qquad \liminf_{\sigma \to \alpha} \frac{\log M(\sigma)}{\{1 - \exp(\sigma - \alpha)\}^{-\rho}} = 0$$

i.e. *the lower type of an analytic function belonging to* D_α *of irreqular growth is zero. And if* (1.5) *holds then*

$$(2.2) \qquad \liminf_{\sigma \to \alpha} \frac{\log m(\sigma)}{\{1 - \exp(\sigma - \alpha)\}^{-\rho}} = 0.$$

Since it follows quite easily by using very elementary arguments, we omit its proof.

Lemma 2. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the class* D_α, *with order* ρ, *lower order* λ $(0 \leq \lambda < \rho < \infty)$. *If* (1.5) *holds ,then*

$$(2.3) \qquad \liminf_{\sigma \to \alpha} \frac{\lambda_{N(\sigma)}}{\{1 - \exp(\sigma - \alpha)\}^{-1-\rho} . \exp(\sigma - \alpha)} = 0.$$

Proof. If

$$\begin{matrix} \mu \\ \delta \end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\lambda_{N(\sigma)}}{\{1 - \exp(\sigma - \alpha)\}^{-1-\rho} . \exp(\sigma - \alpha)},$$

and T, t be respectively the type and lower type of $f(s) \in D_\alpha$, then it can be easily shown that

$$\delta \leq \rho\, t \leq \rho\, T \leq \mu.$$

But by Lemma 1, (since $\lambda < \rho$) $t = o$.

Hence the Lemma follows.

We have seen that $t = 0$ when $\lambda \neq \rho$ so far $0 < \lambda \neq \rho$, we define λ – type of $f(s) \in D_\alpha$ by

$$(2.4) \qquad \liminf_{\sigma \to \alpha} \frac{\log M(\sigma)}{[1 - \exp(\sigma - \alpha)]^{-\lambda}} = t_\lambda .$$

Now we obtain the formula for λ–type in terms of maximum term. For this we need the following Lemma due to Krishna Nandan [1,P.216)]

Lemma 3. *If* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belongs to the class* D_α *and satisfies* (1.5), *then for every* $\gamma^* < \beta$ *and for* σ *sufficiently close to* α

$$(2.5)\quad m(\sigma) < M(\sigma) < m(\sigma)\left[1 + \frac{1+\gamma^*}{\gamma^*} N\left\{\sigma + \frac{1-\exp(\sigma-\alpha)}{N(\sigma)}\right\} x \right.$$

$$\left. x\{1-\exp(\sigma-\alpha)\}^{-1}\right].$$

Theorem 1. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the class* D_α *having lower order* λ $(0 < \lambda < \infty)$. *If* (1.5) *holds, then*

$$(2.6)\qquad \liminf_{\sigma \to \alpha} \frac{\log m(\sigma)}{\{1-\exp(\sigma-a)\}^{-\lambda}} = t_\lambda .$$

Proof. By (2.5), for all σ such that $-\infty < \sigma < \alpha$, we have

$$\log M(\sigma) < \log m(\sigma) + \log\left[1 + \frac{1+\gamma^*}{\gamma^*} N\left\{ + \frac{1-\exp(\sigma-\alpha)}{N(\sigma)}\right\}\right]$$

$$- \log\{1-\exp(\sigma-\alpha)\},$$

and from (1.3) and (1.2),we have for σ sufficiently close to α

$$N(\sigma) < 2.3^{\rho+\varepsilon} \,.\, (D+\varepsilon)\ \{1-\exp(\sigma-\alpha)\}^{-(1+\rho+\varepsilon)}.$$

Dividing both the sides by $\{1-\exp(\sigma-\alpha)\}^{-\lambda}$ and proceeding to limits, we get

$$t_\lambda \le \liminf_{\sigma \to \alpha} \frac{\log m(\sigma)}{\{1-\exp(\sigma-\alpha)\}^{-\lambda}} .$$

Since $m(\sigma) \le M(\sigma)$, the reverse inequalities follow, hence the theorem.

Next we obtain some relations involving type, λ – type and the ma-maximum term.

Theorem 2. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the class* D_α *having order* ρ, *lower order* λ $(0 < \lambda < \rho < \infty)$, *type* T *and* λ*-type* t_λ *and*

$$\limsup_{\sigma \to \alpha} \frac{\lambda_{N(\sigma)}}{\{1 - \exp(\sigma - \alpha)\}^{-1-\rho}. \exp(\sigma-\alpha)} = c$$

$$\liminf_{\sigma \to \alpha} \frac{\lambda_{N(\sigma)}}{\{1 - \exp(\sigma - \alpha)\}^{-1-\lambda} . \exp(\sigma-\alpha)} = d$$

and let

$$T_\rho(\sigma) = \frac{\log m(\sigma)}{\{1-\exp(\sigma - \alpha)\}^{-\rho}}, \quad T_\lambda(\sigma) = \frac{\log m(\sigma)}{\{1 - \exp(\sigma - \alpha)\}^{-\lambda}},$$

then

(2.7) $c-\rho T \leq \limsup_{\sigma \to \alpha} \{1-\exp(\sigma-\alpha)\} T'(\sigma) / \exp(\sigma-x\alpha)\} \leq c,$

(2.8) $-\infty \leq \liminf_{\sigma \to \alpha} \{1-\exp(\sigma-\alpha)\} T'(\sigma) / \exp(\sigma-\alpha) \leq d - \lambda t_\lambda.$

(2.9) $\log m(\sigma) = \log m(\sigma_1) + \int_{\sigma_1}^{\sigma} \lambda_{N(u)} du, \ -\infty < \sigma_1 < \sigma < \sigma.$

Proof. It is known [1, P. 215]

Dividing on both sides of (2.9) by $\{1 - \exp(\sigma-\alpha)\}^{-\rho}$ and then differentiating w.r.t. σ, we get for almost all values $\sigma > \sigma_1$

$$\frac{\{1 - \exp(\sigma-\alpha)\} . T'(\sigma)}{\exp(\sigma-\alpha)} = - \frac{\log m(\sigma_1)}{\{1 - \exp(\sigma-\alpha)\}^{-\rho}} - \frac{\int_{\sigma_1}^{\sigma} \lambda_{N(u)} du}{\{1 - \exp(\sigma - \alpha)\}^{-\rho}} + \frac{\lambda_{N(\sigma)}}{[\{1 - \exp(\sigma-\alpha)\}^{-\rho-1}. \exp(\sigma-\alpha)]}.$$

Proceeding to limits and making use of (2.9) and (2.2), the relation (2.7) follows. Similarly, dividing on both sides of (2.11) by $\{1 - \exp(\sigma-\alpha)\}^{-\lambda}$ and proceeding as above we get (2.8).

3. In this section we start with an arbitrary constant γ and obtain theorems from which results pertaining to λ-type and lower type will

follow immediately. Finally we obtain some relations involving their λ–type and the ratio of the consecutive coefficeients of their Dirichlet series expansion.

Theorem 3. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the class* $D\alpha$ *and* γ $(0 < \gamma < \infty)$ *be an arbitrary number for which*

$$(3.1) \qquad \liminf_{\sigma \to \alpha} \frac{\log M(\sigma)}{\{1 - \exp(\sigma-\alpha)\}^{-\gamma}} = t_\gamma,$$

then

$$(3.2) \qquad \frac{(1+\gamma)^{1+\gamma}}{\gamma^\gamma} \geq \liminf_{n \to \infty} \lambda_{n-1} \left[\log^+\{|a_n| \exp(\alpha\lambda_n)\}^{\frac{1}{\lambda_n}}\right]^{1+\gamma}$$

And, if $\lambda_n \backsim \lambda_{n+1}$, *and* $\Psi(n) = \dfrac{\log \left|\dfrac{a_n}{a_{n+1}}\right|}{\lambda_{n+1}-\lambda_n}$ *forms a non-decreasing function of* n *for* $n > n_o$, *then*

$$(3.3) \qquad \frac{(1+\gamma)^{1+\gamma}}{\gamma^\gamma} t_\gamma = \liminf_{n \to \infty} \lambda_n \left[\log^+\{|a_n| \exp(\alpha\lambda_n)\}^{\frac{1}{\lambda_n}}\right]^{1+\gamma}$$

where (1.5) *holds.*

Our next theorem gives a coefficient characterization of the λ-type which holds for a wider sublcass of functions of the class Dα.

Theorem 4. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the calass* D_α *with* γ $(0 < \gamma < \infty)$ *and satisfy* (1.5). *If* $\{n_k\}_1^\infty$ *be the range of* $N(\sigma)$ *such that* $\lambda_{n_{k-1}} \backsim \lambda_{n_k}$ *as* $k \to \infty$, *then*

$$(3.4) \frac{(1+\gamma)^{1+\gamma}}{\gamma^\gamma} t_\gamma = \max_{\{m_k\}} \left[\liminf_{k \to \infty} \lambda_{m_{k-1}} \{\log^+ \{|a_{m_k}| \exp(a\lambda_{m_k})\}^{\frac{1}{\lambda_{mk}}}\}^{1+\gamma}\right].$$

We omit the proofs of Theorems 3 and 4 ,since these are based on same lines as given by Krishna Nandan ([2]. The main thing that can be pointed out is that one can take an arbitrary number γ $(0 < \gamma < \infty)$ in place of ρ and still the result holds, when $\gamma = \lambda$, we call t_γ to be λ–type of f(s).

Remark. The following results can easily be seen

(i) If $\gamma > \lambda$, then $t_\gamma = 0$.

(ii) If $\gamma < \lambda$, then $t_\gamma = \infty$.

Theorem 5. *Let* $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ *belong to the class* D_α *having order* ρ *and lower order* λ *and* γ $(0 < \gamma < \infty)$ *be an arbitrary number, then*

(3.5) $\frac{1}{\gamma} \left(\frac{\beta+\gamma}{1+\gamma}\right)^{1+\gamma} p \leq t_\gamma$, and

(3.6) $p = 0$ if $\gamma = \rho > \lambda$,

where

$$\beta = \liminf_{n \to \infty} \frac{\lambda_{n-1}}{\lambda_n},$$

and

$$p = \liminf_{n \to \infty} \lambda_{n-1} \left[\log^+ \{(\exp \alpha).|a_n / a_{n-1}|^{\frac{1}{\lambda_n - \lambda_{n-1}}}\right]^{1+\gamma}.$$

Proof. First assume that $0 < p < \infty$. For any $\varepsilon > 0$ such that $p > e > 0$, we have for all $m > N = N(\varepsilon)$,

$$\log |a_m/a_{m-1}| > (\lambda_m - \lambda_{m-1}) \left\{ \left(\frac{p-\varepsilon}{\lambda_{m-1}}\right)^{\frac{1}{1+\gamma}} - \alpha \right\}.$$

Writing the above inequality for $m = N+1, \ldots, n$ and adding all such inequalities, we get

$$\log|a_n| > (p-\varepsilon)^{\frac{1}{1+\gamma}}\Big[\lambda_{n-1}^{-\frac{1}{1+\gamma}} . \lambda_n - \sum_{m=N+1}^{n-1} \lambda_m \Big(\lambda_m^{\frac{1}{1+\gamma}} - \lambda_{m-1}^{\frac{1}{1+\gamma}}\Big)$$

$$- \lambda_N^{\frac{\gamma}{1+\gamma}}\Big] + \log|a_N| - \alpha(\lambda_n - \lambda_N),$$

$$= (p-\varepsilon)^{\frac{1}{1+\gamma}}\Big[\lambda_{n-1}^{-\frac{1}{1+\gamma}} . \lambda_n - \int_{\lambda_N}^{\lambda_{n-1}} n(t)\, d\Big(t^{\frac{1}{1+\gamma}}\Big) - \lambda_N^{\frac{\gamma}{1+\gamma}}\Big] +$$

$$+ \log|a_N| - \alpha(\lambda_n - \lambda_N).$$

where $n(t) = \lambda_m$ for $\lambda_{m-1} \leq t < \lambda_m$. Thus we have

$$\log|a_n| > (p-\varepsilon)^{\frac{1}{1+\gamma}}\Big[\lambda_n . \lambda_{n-1}^{\frac{1}{1+\gamma}} + \frac{1}{\gamma} . \lambda_{n-1}^{\frac{\gamma}{1-\gamma}} - \frac{1+\gamma}{\gamma}\lambda_N^{\frac{\gamma}{1+\gamma}}\Big]$$

$$+ \log|a_N| - \alpha(\lambda_n - \lambda_N) .$$

Thus, on using (3.2), we get

$$\frac{(1+\gamma)^{1+\gamma}}{\gamma^\gamma} t_\gamma \geq (p-\varepsilon)\left(\frac{\gamma+\beta}{\gamma}\right)^{1+\gamma} .$$

Since ε is arbitrary, we have (3.5).

For $\gamma = \rho > \lambda$, we have $t_\gamma = 0$ and hence (3.6) follows with the help of (3.5).

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