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## **Theorems On Fixed Points In Metric Spaces**

By

M.S. KHAN and M.D. KHAN

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Faculté des Sciences de l'Université d'Ankara  
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## Theorems On Fixed Points In Metric Spaces

M. S. KHAN\* and M. D. KHAN\*\*

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### ABSTRACT

Some results concerning fixed points of asymptotically regular maps and asymptotically regular sequences have been established. Our work generalizes well-known fixed point theorems due to Hardy-Rogers and Reich.

### INTRODUCTION

The well-known Banach Contraction Principle states that on a complete metric space  $(X, d)$ , a self-mapping  $T$  for which

$$(A) \quad d(Tx, Ty) \leq k d(x, y),$$

for all  $x, y \in X$ ,  $0 < k < 1$ , always has a unique fixed point. Several generalizations and extensions of this celebrated result have appeared in recent years. For example (Reich [4]), the conclusion of Banach Fixed Point Theorem still holds good when (A) is replaced by

$$d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y),$$

where  $a_i (i = 1, 2, 3)$  are non-negative reals with  $\sum_{i=1}^3 a_i < 1$ .

The fixed point theorem of Reich was further extended by Hardy-Rogers [3] by considering contraction condition involving five control constants whose sum is bounded by 1. In all those results one starts with certain sequences of iterates.

\*DEPARTMENT OF MATHEMATICS ALIGARH MUSLIM UNIVERSITY ALIGARH - 202001 (INDIA).

\*\*CENTRAL SCHOOL, BABINA CANTT. - 284401, Dist. JHANSI (INDIA)

In this note we prove some generalizations of Hardy-Rogers' Theorem to mappings for which certain sequences are asymptotically regular. We also present results on fixed points of asymptotically regular maps.

The technique of our proofs can be used equally well for other results in literature to show that one need not consider the sequence of successive approximation to prove the existence of fixed points of certain mappings.

### Results for Asymptotically Regular Sequences

The following definition is essentially borrowed from Engl [2].  
 Definition 2.1: Let  $(X, d)$  be a metric space. A sequence  $\{x_n\}$  in  $X$  is said to be *asymptotically T-regular* if

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Theorem 2.2: Let  $(X, d)$  be a complete metric space and  $T$  a self-mapping of  $X$  satisfying the inequality:

$$(B) \quad d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\ + a_4 d(y, Tx) + a_5 d(x, y),$$

for all  $x, y \in X$ , where  $a_i (i = 1, 2, 3, 4, 5)$  are non-negative reals and  $\max \{ (a_1 + a_4), (a_3 + a_4 + a_5) \} < 1$ . If there exists an asymptotically T-regular sequence in  $X$ , then  $T$  has a unique fixed point.  
 Proof: Let  $\{x_n\}$  be an asymptotically T-regular sequence in  $X$ .

Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, Tx_n) + d(Tx_n, x_m) \\ &\leq d(x_n, Tx_n) + d(Tx_n, Tx_m) + d(Tx_m, x_m) \\ &\leq d(x_n, Tx_n) + d(Tx_m, x_m) \\ &\quad + \{ a_1 d(x_n, Tx_n) + a_2 d(x_m, Tx_m) \\ &\quad + a_3 d(x_n, Tx_m) + a_4 d(x_m, Tx_n) + a_1 d(x_n, x_m) \} \\ &\leq d(x_n, Tx_n) + d(Tx_m, x_m) + \{ a_1 d(x_n, Tx_n) \\ &\quad + a_2 d(x_m, Tx_m) + a_3 d(x_n, x_m) + a_3 d(x_m, Tx_m) \\ &\quad + a_4 d(x_m, x_n) + a_4 d(Tx_n) + a_5 d(x_n, x_m) \}. \end{aligned}$$

Thus we get

$$d(x_n, x_m) \leq \left( \frac{1+a_1+a_4}{1-a_3-a_4-a_5} \right) d(x_n, Tx_n) + \left( \frac{1+a_2+a_3}{1-a_3-a_4-a_5} \right) d(x_m, Tx_m)$$

Taking limit as  $n$  tends to infinity, we have  $\lim_{n \rightarrow \infty} d(x_n, x_m) = 0$

showing thereby that  $\{x_n\}$  is a Cauchy sequence. Since  $X$  is complete, put  $\lim_{n \rightarrow \infty} x_n = z$  (say). Now we claim that  $z$  is a fixed point of  $T$ . To do this consider,

$$\begin{aligned} d(Tz, z) &\leq d(Tz, Tx_n) + d(Tx_n, x_n) + d(x_n, z) \\ &\leq a_1 d(z, Tz) + a_2 d(x_n, Tx_n) + a_3 d(z, Tx_n) \\ &\quad + a_4 d(x_n, Tz) + a_5 d(z, x_n) \\ &\quad + d(Tx_n, x_n) + d(x_n, z) \\ &\leq a_1 d(z, Tz) + a_2 d(x_n, Tx_n) + a_3 d(z, x_n) \\ &\quad + a_3 d(x_n, Tx_n) + a_4 d(x_n, z) + a_4 d(z, Tz) \\ &\quad + a_5 d(z, x_n) + d(Tx_n, x_n) + d(x_n, z). \end{aligned}$$

Therefore,

$$(1-a_1-a_4) d(Tz, z) \leq (1+a_2+a_3) d(x_n, Tx_n) + (1+a_3+a_4+a_5) d(x_n, z) \text{ which gives}$$

$$d(Tz, z) \leq \left( \frac{1+a_2+a_3}{1-a_1-a_4} \right) d(x_n, Tx_n) + \left( \frac{1+a_3+a_4+a_5}{1-a_1-a_4} \right) d(x_n, z).$$

Since  $T$  is asymptotically  $T$ -regular, taking limit as  $n \rightarrow \infty$  we are left with  $d(Tz, z) = 0$ , i.e.  $Tz = z$ . Hence  $z$  is a fixed point of  $T$ .

To show the uniqueness, let  $z \neq z_1$  be two fixed points, then

$$\begin{aligned} d(z, z_1) &= d(Tz, Tz_1) \\ &\leq a_1 d(z, Tz) + a_2 d(z_1, Tz_1) \\ &\quad + a_3 d(z, Tz_1) + a_4 d(z, Tz) + a_5 d(z, z_1) \end{aligned}$$

$(1-a_3-a_4-a_5) d(z, z_1) = 0$ , whence uniqueness follows immediately.

This completes the proof.

If  $T$  is continuous, then existence part follows very easily as shown by the following theorem. In this case condition (B) is not needed.

Theorem 2.3. Let  $(X, d)$  be a metric space and  $T$  a continuous self-mappings of  $X$ . If there exists an asymptotically  $T$ -regular sequence  $\{x_n\}$  with  $\lim_{n \rightarrow \infty} x_n = z$ , then  $z$  is a fixed point of  $T$ .

Proof: Consider,

$$d(Tz, z) \leq d(Tz, Tx_n) + d(Tx_n, x_n) + d(x_n, z)$$

Then taking limit as  $n \rightarrow \infty$ , we have

$$d(Tz, z) = 0$$

So  $Tz = z$ . Hence  $z$  is a fixed point of  $T$ .

### Results for Asymptotically Regular Maps.

Following Browder and Petryshyn [1] we have the following: Definition 3.1. Let  $(X, d)$  be a metric space. A mapping  $T$  of  $X$  into itself is said to be *asymptotically regular* at a point  $x$  in  $X$  if

$$\lim_{n \rightarrow \infty} d(T^n x, T^{n+1} x) = 0.$$

Theorem 3.2. Let  $(X, d)$  be a complete metric space and  $T$  a self mapping of  $X$  satisfying the inequality

$$\begin{aligned} d(Tx, Ty) &\leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\ &\quad + a_4 d(y, Tx) + a_5 d(x, y), \end{aligned}$$

for all  $x, y \in X$ , where  $a_i (i = 1, 2, 3, 4, 5)$  are non-negative reals with  $\max \{(a_1 + a_4), (a_3 + a_4 + a_5)\} < 1$ . If  $T$  is asymptotically regular at some point  $x$  of  $X$ , then there exist a unique fixed point of  $T$ .

Proof: Let  $T$  be an asymptotically regular at  $x_0 \in X$ . Consider the sequence  $\{T^n x_0\}$ , then for all  $m, n \geq 1$

$$\begin{aligned} d(T^m x_0, T^n x_0) &\leq a_1 d(T^{m-1} x_0, T^m x_0) + a_2 d(T^{n-1} x_0, T^n x_0) \\ &\quad + a_3 d(T^{m-1} x_0, T^n x_0) + a_4 d(T^{n-1} x_0, T^m x_0) \\ &\quad + a_5 d(T^{m-1} x_0, T^{n-1} x_0) \\ &\leq a_1 d(T^{m-1} x_0, T^m x_0) + a_2 d(T^{n-1} x_0, T^n x_0) \\ &\quad + a_3 d(T^{m-1} x_0, T^m x_0) + a_3 d(T^m x_0, T^n x_0) \\ &\quad + a_4 d(T^{n-1} x_0, T^n x_0) + a_4 d(T^n x_0, T^m x_0) \\ &\quad + a_5 d(T^{m-1} x_0, T^m x_0) + a_5 d(T^m x_0, T^n x_0) \\ &\quad + a_5 d(T^n x_0, T^{n-1} x_0). \end{aligned}$$

Hence we get

$$\begin{aligned} d(T^m x_0, T^n x_0) &\leq \left( \frac{a_1 + a_3 + a_5}{1 - a_3 - a_4 - a_5} \right) d(T^{m-1} x_0, T^m x_0) \\ &\quad + \left( \frac{a_2 + a_4 + a_5}{1 - a_3 - a_4 - a_5} \right) d(T^{n-1} x_0, T^n x_0). \end{aligned}$$

Since  $T$  is asymptotically regular, as  $m, n \rightarrow \infty$ , above yields  $\lim_{n \rightarrow \infty} d(T^m x_0, T^n x_0) = 0$ .

This shows that  $\{T^n x_0\}$  is a Cauchy sequence, since  $X$  is complete,  $\lim_{n \rightarrow \infty} T^n x_0 = z$ .

Now we claim that  $z$  is fixed point of  $T$ . For this we consider  $d(Tz, z) \leq d(Tz, T^n x_0) + d(T^n x_0, z)$

$$\begin{aligned} &\leq a_1 d(z, Tz) + a_2 d(T^{n-1} x_0, T^n x_0) \\ &\quad + a_3 d(z, T^n x_0) + a_4 d(T^{n-1} x_0, Tz) \\ &\quad + a_5 d(z, T^{n-1} x_0) + d(T^n x_0, z) \\ &\leq a_1 d(z, Tz) + a_2 d(T^{n-1} x_0, T^n x_0) + a_3 d(z, T^n x_0) \\ &\quad + a_4 d(T^{n-1} x_0, T^n x_0) + a_4 d(T^n x_0, Tz) \\ &\quad + a_5 d(z, T^n z) + a_5 d(T^n z, T^{n-1} x_0) \\ &\quad + d(T^n x_0, z). \end{aligned}$$

Letting  $n$  tending to infinity, we get

$$d(Tz, z) \leq a_1 d(z, Tz) + a_4 d(z, Tz) \text{ giving}$$

$d(Tz, z) = 0$ . Hence  $Tz = z$ . Therefore  $z$  is a fixed point of  $T$ . The unicity of fixed point  $z$  follows as in Thorem 2.2.

**Theorem 3.3.** Let  $(X, d)$  be a metric space and  $T$  a selfmapping satisfying the inequality,

$$\begin{aligned} d(Tx, Ty) &\leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) \\ &\quad + a_1 d(y, Tx) + a_1 d(x, y) \end{aligned}$$

for all  $x, y \in X$ , where  $a_i (i = 1, 2, 3, 4, 5)$  are non-negative reals with  $\max \{ (a_2 + a_3), (a_3 + a_4 + a_5) \} < 1$ . If  $T$  is asymptotically regular at some point  $x$  in  $X$  and the sequence  $\{T^n x\}$  of iterates has a subse-

quence  $\{T^n_k x\}$  converging to a point  $z$  of  $X$ , then  $z$  is a unique fixed point of  $T$  and  $\{T^n x\}$  also converges to  $z$ .

Proof: Let  $\lim_k T^n_k x = z \neq Tz$ , then

$$\begin{aligned} d(z, Tz) &\leq d(z, T^n_k x) + d(T^n_k x, T^{n_k+1} x) + d(T^{n_k+1} x, Tz) \\ &\leq d(z, T^n_k x) + d(T^n_k x, T^{n_k+1} x) \\ &\quad + a_1 d(T^n_k x, T^{n_k+1} x) + a_2 d(z, Tz) \\ &\quad + a_3 d(T^n_k x, Tz) + a_4 d(z, T^{n_k+1} x) \\ &\quad + a_5 d(T^n_k x, z). \end{aligned}$$

Then  $n \rightarrow \infty$  yields that

$$d(z, Tz) \leq (a_2 + a_3) d(z, Tz),$$

whence  $z$  is a fixed point of  $T$ . Now,

$$\begin{aligned} d(z, T^n x) &= d(Tz, T^n x) \\ &\leq a_1 d(z, Tz) + a_2 d(T^{n-1} x, T^n x) \\ &\quad + a_3 d(z, T^n x) + a_4 d(T^{n-1} x, Tz) \\ &\quad + a_5 d(z, T^{n-1} x) \\ &\leq a_1 d(z, Tz) + a_2 d(T^{n-1} x, T^n x) \\ &\quad + a_3 d(z, T^n x) + a_4 d(T^{n-1} x, T^n x) \\ &\quad + a_4 d(T^n x, Tz) + a_5 d(z, T^n x) \\ &\quad + a_5 d(T^n x, T^{n-1} x). \end{aligned}$$

So,

$$(1 - a_3 - a_4 - a_5) d(z, T^n x) \leq (a_2 + a_4 + a_5) d(T^{n-1} x, T^n x).$$

Asymptotic regularity of  $T$  at  $x$  and the fact  $a_3 + a_4 + a_5 < 1$  imply that the sequence  $\{T^n x\}$  converges to  $z$ . This completes the proof.

Remarks (i). It is clear that the asymptotic regularity of the mapping  $T$  satisfying Hardy-Rogers' contraction condition is actually a consequence of

of  $\sum_{i=1}^5 a_i < 1$ . So our Theorem 3.2 and Theorem 3.3 extend results due to Hardy-Rogers [3]. It is also worth mentioning

that our condition on control constants says that  $\sum_{i=1}^5 a_i$  may exceed 1

(ii). Recently, Smart [5] has given an example to the effect that asymptotic regularity of a mapping need not imply the convergence of its sequence of iterates.

(iii). The following example shows that if  $T$  is not asymptotically regular at any point of the space, then all other conditions of Theorem 3.2 are not sufficient to ensure the existence of a fixed point of  $T$ .

Example: Let us consider the complete metric space  $X = \{0\} \cup [1, \infty]$  with the metric  $d(x, y) = |x - y|$ ,  $x, y \in X$ , and let  $T$  be the non-contraction mapping of  $X$  into itself defined by

$$Tx = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then taking  $a_1 = 0$ ,  $a_2 = \frac{1}{4}$ ,  $a_3 = \frac{1}{2}$ ,  $a_4 =$  any positive real number and  $a_5 = 0$ , we see that  $T$  satisfies Hardy-Rogers's contraction condition but  $T$  is not asymptotically regular at any point of  $X$ . Further,  $T$  leaves no point fixed.

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