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On Means Of Entire Functions With Index-Pair (p,q)

by

H. S. KASANA

16

Faculté des Sciences de l'Université d'Ankara
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On Means Of Entire Functions With Index-Pair (p,q)

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H. S. KASANA

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ABSTRACT

Here, we introduce the generalized mean function $m_{\delta,k}$ for entire functions represented by Dirichlet series with index-pair (p,q). Besides, studying the relative growth of this mean with respect to the fundamental mean I_δ, we have derived some formulae for (p, q)-orders and (p, q)-types in terms of I_δ and $m_{\delta,k}$ which are extensions and improvements of many of the known results.

1. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $s = \sigma + it$, $o \leq \lambda_1 < \lambda_n < \lambda_{n+1} \to \infty$ as $n \to \infty$, be an entire Dirichlet series. The concept of (p, q)- order, lower (p, q)-order, (p, q)-type and lower (p, q)-type of f(s) having index-pair (p, q), $p \geq q + 1 \geq 1$, has recently been introduced by Juneja et al. ([6], [7]).

Let δ, k be any positive real numbers and define

$$(1.1)\quad I_\delta(\sigma) = \left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^\delta \right\}^{1/\delta}$$

In order to study the growth properties of entire Dirichlet series of Simple Ritt-order Kamthan [8] defined

$$(1.2)\quad m'_{,\delta k}(\sigma) = \frac{1}{e^{k\sigma}} \int_{0}^{\sigma} I_\delta(x)\, e^{kx}\, dx.$$

Again, to study the analogous results for entire Dirichlet series of slow growth i. e., (2, 1)- order, Jain and Chugh [5], introduced the following mean

$$(1.3)\qquad m^*_{\delta,k}(\sigma) = \frac{1}{\sigma^{k+1}} \int_0^{\sigma} I_\delta(x)\, x^k dx.$$

Later on Jain [4] also defined $N_{\delta,k}(\sigma)$ as

$$(1.4)\qquad N_{\delta,k}(\sigma) = \exp\left\{ \frac{1}{e^{k\sigma}} \int_0^{\sigma} \log I_\delta(x)\, e^{kx}\, dx \right\}.$$

Now it becomes a naturel question to introduce the most generalized mean in context to the recently developed growth parameters such as (p, q)-orders and (p, q)-types. We shall term the generalized mean as auxiliary mean to $I_\delta(\sigma)$ and define

$$(1.5)\ m_{\delta,k}(\sigma) = \exp^{[p-2]}\left[\frac{1}{(\log^{[q-1]}\sigma)^k} \int_{\sigma_0}^{\sigma} \frac{\log^{[p-2]} I_\delta(x)\,(\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\, dx\right]$$

where $\log^{[p]}x$ denoted the pth iterate of $\log x$, $\Lambda_{[q]}(x) = \prod_{i=0}^{q} \log^{[i]}x$,

$\exp^{[p]} x = \log^{[-p]}x$ and $\sigma_0 = \exp^{[q-2]}[1]$

Doherey and Srivastava [1] has shown that for an entire Dirichlet series of (p, q)-order ρ and lower (p, q) -order λ

$$(1.6)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p]} I_\delta(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho(p,q) \equiv \rho\\ \lambda(p,q) \equiv \eta\end{matrix}.$$

Following Kamthan [9] it can be proved that for an entire Dirichlet series of (p, q)-order ρ $(b < \rho < \infty)$, (p, q)-type τ and lower (p, q)-type ν

$$(1.7)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} I_\delta(\sigma)}{(\log^{[p-1]}\sigma)^{\rho}} = \begin{matrix}\tau(p,q) \equiv \tau\\ \nu(p,q) \equiv \nu\end{matrix}$$

where $b = 1$ if $p = q + 1$ and $b = 0$ if $p > q + 1$.

In this paper, we give some properties of the auxiliary mean defined in (1.5). We have studied relative growth of this mean to the fundamental mean $I_\delta(\sigma)$. Here, we are restricted to deal with a class of entire Dirichlet series with index-pair (p, q) for which $\log^{[p-1]} I_\delta(\sigma)$ is an increasing convex function of $\log^{[q]}\sigma$.

2. We first prove a few lemmas which will be used in the sequel:

Lemma 1. If φ, Ψ and $\frac{\varphi'}{\Psi'}$ are positive increasing functions of σ for $\sigma > \sigma_0$ and if $\varphi(\sigma_0) = \Psi(\sigma_0) = 0$, then $\frac{\varphi}{\Psi}$ is an increasing function of σ for $\sigma > \sigma_0$

Proof. Its proof is due to Hardy, Littlewood and Polya [2].

Lemma 2. $(\log^{[q-1]}\sigma)^k \log^{[p-2]} I_\delta(\sigma)$ is an increasing convex function of $(\log^{[q-1]}\sigma)^k \log^{[p-2]} m_{\delta,k}(\sigma)$ for $\sigma > \sigma_0$.

Proof. We have

$$\frac{d\,[(\log^{[q-1]}\sigma)^k \log^{[p-2]} I_\delta(\sigma)]}{d\,[(\log^{[q-1]}\sigma)^k \log^{[p-2]} m_{\delta,k}(\sigma)]}$$

$$= \frac{\frac{d}{d\sigma}\,[(\log^{[q-1]}\sigma)^k \log^{[p-2]} I_\delta(\sigma)]}{\frac{d}{d\sigma}\,[(\log^{[q-1]}\sigma)^k \log^{[p-2]} m_{\delta,k}(\sigma)]}$$

$$= \frac{\dfrac{k(\log^{[q-1]}\sigma)^{k-1}}{\Lambda_{[q-2]}(\sigma)} \log^{[p-2]} I_\delta(\sigma) + \dfrac{(\log^{[q-1]}\sigma)^k I'_\delta(\sigma)}{\Lambda_{[p-2]}(I_\delta(\sigma))}}{\dfrac{(\log^{[q-1]}\sigma)^{k-1} \log^{[p-2]} I_\delta(\sigma)}{\Lambda_{[q-2]}(x)}}$$

$$= \left[k + \frac{I'_\delta(\sigma)\,\Lambda_{[q-1]}(\sigma)}{\Lambda_{[p-2]}(I_\delta(\sigma))}\right]$$

By assumption, $\log^{[p-1]} I_\delta(\sigma)$ is an increasing convex function of $\log^{[q]}\sigma$, the quantity inside the bracket is an increasing function of σ for $\sigma > \sigma_0$ and hence the lemma.

Lemma 3. $\log^{[p-2]} I_\delta(\sigma) / \log^{[p-2]} m_{\delta,k}(\sigma)$ is an increasing function of σ for $\sigma > \sigma_0$.

Proof. This is a direct consequence of Lemma 1 and Lemma 2.

Lemma 4. $\log^{[p-1]} m_{\delta,k}(\sigma)$ is an increasing convex function of $\log^{[q]} \sigma$ for $\sigma > \sigma_0$.

Proof. We have

$$\frac{d\,[\log^{[p-1]} m_{\delta,k}(\sigma)]}{d\,[\log^{[q]}\sigma]} = \frac{\dfrac{d}{d\sigma}\,[\log^{[p-1]} m_{\delta,k}(\sigma)]}{\dfrac{d}{d\sigma}\,[\log^{[q]}\sigma]}$$

$$= -k + \frac{\log^{[p-2]} I_\delta(\sigma)}{\log^{[p-2]} m_{\delta,k}(\sigma)}.$$

Using lemma 4, we conclude that

$$\frac{d^2\,[\log^{[p-1]} m_{\delta,k}(\sigma)]}{d\,[\log^{[q]}\sigma]^2} > 0 \quad \text{for } \sigma > \sigma_0,$$

and hence the lemma.

We now prove

Theorem 1. For an entire Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ with index-pair (p, q), (p, q)-order ρ and lower (p, q)-order λ, we find that

$$(2.1 \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}.$$

Proof. Since $\log^{[p-2]} I_\delta(\sigma)$ is an increasing function of σ for $\sigma > \sigma_0$, we observe that

$$\log^{[p-2]} m_{\delta,k}(\sigma) = \frac{1}{(\log^{[q-1]}\sigma)^k} \int_{\sigma_0}^{\sigma} \frac{\log^{[p-2]} I_\delta(x)\,(\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\,dx$$

$$< \frac{\log^{[p-2]} I_\delta(\sigma)}{(\log^{[q-1]}\sigma)^k} \int_{\sigma_0}^{\sigma} \frac{(\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\,dx$$

$$\simeq \frac{1}{k} \log^{[p-2]} I_\delta(\sigma) \{1 + o(1)\}.$$

Hence,

$$(2.2) \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\log^{[p]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} \leq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\log^{[p]} I_\delta(\sigma)}{\log^{[q]}\sigma}.$$

Further,

$$\log^{[p-2]} m_{\delta,k}(\sigma') = \frac{1}{(\log^{[q-1]}\sigma')} \int_{\sigma_0}^{\sigma'} \frac{\log^{[p-2]} I_\delta(x) \log(^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)} dx$$

where $\sigma' = \exp^{[q-1]} \{(\log^{[q-1]}\sigma)^k + d\}^{1/k} > \sigma > \sigma_0, d > 0.$

Therefore,

$$\log^{[p-2]} m_{\delta,k}(\sigma') > \frac{\log^{[p-2]} I_\delta(\sigma)}{(\log^{[q-1]}\sigma')^k} \int_{\sigma}^{\sigma'} \frac{\log^{[q-1]} x)^{k-1}}{\Lambda_{[q-2]}(x)} dx$$

$$= \frac{d}{k} \cdot \frac{\log^{[p-2]} I_\delta(\sigma)}{(\log^{[q-1]}\sigma')^k}$$

or,

$$\log^{[p-1]} m_{\delta,k}(\sigma') > 0(1) + \log^{[p-1]} I_\delta(\sigma) - k \log^{[q]} \sigma'.$$

On using the definition of index-pair and relation (1.6), we get

$$\log^{[p-1]} m_{\delta,k}(\sigma') > \log^{[p-1]} I_\delta(\sigma) \{1 + o(1)\}.$$

Finally, we have

$$\frac{\log^{[p]} m_{\delta,k}(\sigma')}{\log^{[q]}\sigma'} > \frac{\log^{[p]} I_\delta(\sigma)}{\log^{[q]}\sigma} \cdot \frac{\log^{[q]}\sigma}{\log^{[q]}\sigma'} + o(1).$$

Since $\log^{[q]}\sigma \simeq \log^{[q]}\sigma'$ as $\sigma \mapsto \infty$, on taking limits in above inequality we get

$$(2.3) \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\log^{[p]} m_{\delta,k}(\sigma)}{\log^{[q]}\sigma} \geq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \\ \inf\end{matrix} \frac{\log^{[p]} I_\delta(\sigma)}{\log^{[q]}\sigma}.$$

Combining (2.2) and (2.3) and taking into account (0.6) the theorem follows.

Theorem 2. For an entire function of (p, q) -order ρ and lower (p, q)-order λ, we have

$$(2.4)\quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{ \frac{\log^{[p-2]} I_\delta(\sigma)}{\log^{[p-2]} m_{\delta,k}(\sigma)} \right\}^{1/\log^{[q]}\sigma} = \begin{matrix} e^{\rho} \\ e^{\lambda} \end{matrix}.$$

Proof. It is readily seen from definition of $m_{\delta,k}(\sigma)$ that

$$\frac{d}{d\sigma}\left[\log\left\{(\log^{[q-1]}\sigma)^k \log^{[p-2]} m_{\delta,k}(\sigma)\right\}\right] = \frac{\log^{[p-2]} I_\delta(\sigma)}{\Lambda_{[q-1]}(\sigma)\log^{[p-2]} m_{\delta,k}(\sigma)}.$$

On integration, we have

$$k\log^{[q]}\sigma + \log^{[p-1]} m_{\delta,k}(\sigma) = 0\,(1) + \int_{\sigma_0}^{\sigma} \frac{\log^{[p-2]} I_\delta(\sigma)}{\log^{[p-2]} m_{\delta,k}(\sigma)\,\Lambda_{[q-1]}(\sigma)}$$

or

$$(2.5)\quad \log^{[p-1]} m_{\delta,k}(\sigma) = 0\,(1) + \int_{\sigma_0}^{\sigma} \frac{\varphi(x)}{\Lambda_{[q-1]}(x)}\,dx,$$

where

$$(2.6)\quad \varphi(x) = \frac{\log^{[p-2]} I_\delta(x)}{\log^{[p-2]} m_{\delta,k}(x)} - k$$

is an increasing function of x for $x > x_0$ (by virtue of Lemma 3). Thus (2.5) gives

$$\log^{[p-1]} m_{\delta,k}(\sigma) < 0\,(1) + \varphi(\sigma)(\log^{[q-1]}\sigma)\{1 + o\,(1)\}.$$

On using Theorem 1, we get from above inequality

$$(2.7)\quad \rho \le \limsup_{\sigma\to\infty} \frac{\log\varphi(\sigma)}{\log^{[q]}\sigma}, \qquad \lambda \le \liminf_{\sigma\to\infty} \frac{\log\varphi(\sigma)}{\log^{[q]}\sigma}.$$

Again,

$$\log^{[p-1]} m_{\delta,k}(\sigma') > \int_{\sigma}^{\sigma'} \frac{\varphi(x)}{\Lambda_{[q-1]}(x)}\,dx,$$

where $\sigma' = \exp^{[q]}(\alpha + \log^{[q]}\sigma) > \sigma, \quad \alpha > 0$.

Hence,

$$\log^{[p-1]} m_{\delta,k}(\sigma') > \varphi(\sigma) \cdot \alpha,$$

which gives,

$$(2.8) \quad \rho \geq \lim_{\sigma\to\infty} \sup \frac{\log \varphi(\varphi)}{\log^{[q]}\sigma}, \qquad \lambda \geq \lim_{\sigma\to\infty} \inf \frac{\log \varphi(\sigma)}{\log^{[q]}\sigma}.$$

Combining (2.7) and (2.8), we get

$$(2.9) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \varphi(\sigma)}{\log^{[q]}\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}.$$

The theorem now follows from (2.6) and (2.9).

Corollary 1. If f(s) is an entire function with index-pair (p,q) then

$$(2.10) \qquad \log^{[p-1]} I_\delta(\sigma) \simeq \log^{[p-1]} m_{\delta,k}(\sigma) \quad \text{as } \sigma \to \infty.$$

Proof. From (2.4), we have for given $\varepsilon > 0$ and $\sigma > \sigma_0$

$$\left\{ \frac{\log^{[p-2]} I_\delta(\sigma)}{\log^{[p-2]} m_{\delta,k}(\sigma)} \right\}^{1/\log^{[q]}\sigma} < e^\rho + \varepsilon$$

or,

$$\frac{\log^{[p-1]} I_\delta(\sigma)}{\log^{[p-1]} m_{\delta,k}(\sigma)} - 1 < \frac{(\rho + \varepsilon)\log^{[q]}\sigma}{\log^{[p-1]} m_{\delta,k}(\sigma)}.$$

Taking limit and using Lemma 4, we get

$$(2.11) \quad \lim_{\sigma\to\infty} \sup \frac{\log^{[p-1]} I_\delta(\sigma)}{\log^{[p-1]} m_{\delta,k}(\sigma)} \leq 1.$$

Similarly taking into consideration the limit infimum in (2.4), we have for any $\varepsilon > 0$ and $\sigma > \sigma_0$,

$$\left\{ \frac{\log^{[p-1]} I_\delta(\sigma)}{\log^{[p-1]} m_{\delta,k}(\sigma)} \right\}^{1/\log^{[q]}\sigma} > e^{\lambda-\varepsilon}$$

and proceeding like above, we reach at

$$(2.12) \quad \lim_{\sigma\to\infty} \inf \frac{\log^{[p-1]} I_\delta(\sigma)}{\log^{[p-1]} m_{\delta,k}(\sigma)} \geq 1.$$

Now, (2.11) and (2.12) together prove the theorem .

Corollary 2. If f (s) is an entire function of (p, q)-orderρ ($b < \rho < \infty$), (p, q)-type τ and lower (p, q)- type ν, then

(2.13) $$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} m_{\delta,k}(\sigma)}{(\log^{[q-1]}\sigma)^p} = \begin{matrix}\tau\\ \nu\end{matrix}.$$

Remarks (i) Theorem 2 includes following results as particular cases:

(a) $(p,q) = (2,0)$, $\delta = 2$; due to Kamthan [8]

(b) $(p,q) = (2,1)$, due to Jain and Chug [5]

(ii) All the results proved in Theorems 1 and 2 also hold for (p,q)-orders and (p,q)-types of entire Taylor series subject to the condition p and q are integers such that $p \geq q \geq 1$. In this case for $(p,q) = (2,1)$, our Theorem 2 includes the results of Rahman [12],

Lakshminarasimhan [10], Polya and Szego [11], Shah [13] as particular cases.

(iii) The following means called aritmetic mean function and auxiliary arithmetic mean function give all the results derived in this paper:

(2.14) $$\mu_\delta(\sigma) = \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\mathrm{Re}\ f(s)|^\delta \right\}^{1/\delta}$$

and

(2.15) $$M_{\delta,k}(\sigma) = \exp^{[p-2]} \left\{ \frac{1}{(\log^{[q-1]}\sigma)^k} \int_{\sigma_0}^{\sigma} \frac{og^{[p-2]} m_\delta(x)\,(\log^{[q-1]}x)^{k-1}}{\Lambda_{[q-2]}(x)}\, dx \right\}$$

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Department of Mathematics,
University of Roorkee,
Roorkee-247667 (INDIA).